

Asymptotically optimal discretization of hedging strategies with jumps

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Abstract

In this work, we consider the hedging error due to discrete trading in models with jumps. Extending an approach developed by Fukasawa (2011) for continuous processes, we propose a framework enabling to (asymptotically) optimize the discretization times. More precisely, a discretization rule is said to be optimal if for a given cost function, no strategy has (asymptotically, for large cost) a lower mean square discretization error for a smaller cost. We focus on discretization rules based on hitting times and give explicit expressions for the optimal rules within this class.

Key words: Discretization of stochastic integrals, asymptotic optimality, hitting times, option hedging, semimartingales with jumps, Blumenthal-Gettoor index

2010 Mathematics Subject Classification: 60H05, 91G20

1 Introduction

A basic problem in mathematical finance is to replicate a random claim with \mathcal{F}_T -measurable payoff H_T with a portfolio involving only the underlying asset Y and cash. When Y follows a diffusion process of the form

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t, \quad (1)$$

it is known that under minimal assumptions, a random payoff depending only on the terminal value of the asset $H_T = H(Y_T)$ can be replicated with the so-called delta hedging strategy. This means that the number of units of underlying to hold at time t is equal to $X_t = \frac{\partial P(t, Y_t)}{\partial Y}$, where $P(t, Y_t)$ is the price of the option, which is uniquely defined in such a model. However, to implement such a strategy, the hedging portfolio must be readjusted continuously, which is

of course physically impossible and anyway irrelevant because of the presence of microstructure effects and transaction costs. For this reason, the optimal strategy is always replaced with a piecewise constant one, leading to a discretization error. The relevant questions are then: (i) how big is this discretization error and (ii) when is the good time to readjust the hedge.

Assume first that the hedging portfolio is readjusted at regular intervals of length $h = \frac{T}{n}$. A result by Ruotao Zhang [19], see also [2, 14] then shows that for Lipschitz continuous payoff functions, assuming zero interest rates, the discretization error

$$\mathcal{E}_T^n = \int_0^T X_t dY_t - \int_0^T X_{h[t/h]} dY_t$$

satisfies

$$\lim_{h \rightarrow 0} n E[(\mathcal{E}_T^n)^2] = \frac{T}{2} E \left[\int_0^T \left(\frac{\partial^2 P}{\partial Y^2} \right)^2 \sigma(s, Y_s)^4 ds \right]. \quad (2)$$

Of course, it is intuitively clear that readjusting the portfolio at regular deterministic intervals is not optimal. However, the optimal strategy for fixed n is very difficult to compute.

Fukasawa [11] simplifies this problem by assuming that the hedging portfolio is readjusted at high frequency. The performance of different discretization strategies can then be compared based on their asymptotic behavior as the number of readjustment dates n tends to infinity, rather than the performance for fixed n . Consider a discretization rule : a sequence of discretization strategies

$$0 = T_0^n < T_1^n < \dots < T_j^n < \dots,$$

with $\sup_j |T_{j+1}^n - T_j^n| \rightarrow 0$ as $n \rightarrow \infty$ and let $N_T^n := \max\{j \geq 0; T_j^n \leq T\}$ be the total number of readjustment dates on the interval $[0, T]$ for given n . To compare different discretization strategies in terms of their asymptotic behavior, Fukasawa [11] uses the criterion

$$\lim_{n \rightarrow \infty} E[N_T^n] E[\langle \mathcal{E}^n \rangle_T], \quad (3)$$

where $\langle \mathcal{E}^n \rangle$ is the quadratic variation of the semimartingale $(\mathcal{E}_t^n)_{t \geq 0}$. He finds that when the underlying asset is a continuous semimartingale, the functional (3) admits a nonzero lower bound over all discretization rules, and exhibits a specific rule which attains this lower bound and is therefore called *asymptotically efficient*.

In the diffusion model (1), the asymptotically efficient rule takes the form

$$T_{j+1}^n = \inf\{t > T_j^n; |X_t - X_{T_j^n}|^2 \geq h_n \frac{\partial^2 P(T_j^n, Y_{T_j^n})}{\partial Y^2}\}, \quad X_t = \frac{\partial P(t, Y_t)}{\partial Y}, \quad (4)$$

where h_n is a deterministic sequence with $h_n \rightarrow 0$. For this efficient rule one has

$$\lim_{n \rightarrow \infty} E[N_T^n] E[\langle \mathcal{E}^n \rangle_T] = \frac{1}{6} E \left[\int_0^T \frac{\partial^2 P}{\partial Y^2} \sigma(s, Y_s)^2 ds \right]^2,$$

whereas for readjustment at equally spaced dates, formula (2) yields

$$\lim_{n \rightarrow \infty} E[N_T^n] E[\langle \mathcal{E}^n \rangle_T] = \frac{T}{2} E \left[\int_0^T \left(\frac{\partial^2 P}{\partial Y^2} \right)^2 \sigma(s, Y_s)^4 ds \right],$$

Using the Cauchy-Schwarz inequality, we then see that the asymptotically efficient strategy leads to a gain of at least a factor 3 (and in practice, much more), compared to readjustment at regularly spaced points.

While the above approach is quite natural and provides very explicit results, it fails to take into account important factors of market reality. First, the asymptotic functional (3) is somewhat ad hoc, and does not reflect any specific model for the transaction costs. Yet, transaction costs are one of the main reasons why continuous (or almost continuous) readjustments are not used. Therefore, they should be the determining factor for any discretization algorithm. On the other hand, the continuity assumption, especially at relatively high frequencies, is not realistic. Indeed, it is well known that jumps in the price occur quite frequently and have a significant impact on the hedging error. It can even be argued that high-frequency financial data are best described by pure jump processes [6].

The objective of this paper is therefore two-fold. First, we develop a framework for characterizing the asymptotic efficiency of discretization strategies which takes into account the transaction costs. Second, we remove the continuity assumption in order to understand the effect of the activity of small jumps (often quantified by the Blumenthal-Gettoor index) on the optimal discretization strategies.

Models with jumps correspond to incomplete markets, where hedging is an approximation problem:

$$\min_X E \left(c + \int_0^T X_{t-} dY_t - H_T \right)^2, \quad (5)$$

where Y is now a semimartingale with jumps. The optimal strategy X^* for this problem is known to exist for any $H_T \in L^2$ [10, 9, 18, 16, 7]. If the expectation in (5) is computed under a martingale probability measure, then for any admissible

strategy X' ,

$$E \left(c + \int_0^T X'_{t-} dY_t - H_T \right)^2 = E \left(\int_0^T (X'_{t-} - X^*_{t-}) dY_t \right)^2 + E \left(c + \int_0^T X^*_{t-} dY_t - H_T \right)^2. \quad (6)$$

Indeed, $\int X^*_{t-} dY_t$ is essentially the orthogonal projection of H_T on the subspace of L^2 constituted by the stochastic integrals of the form $\int X_{t-} dY_t$ where X_{t-} is an admissible hedging strategy. Therefore, the quadratic hedging problem (5) and the discretization problem can be studied separately. Given that the quadratic hedging problem has already been studied by many authors, in this paper we concentrate on the discretization problem.

Our goal is to study and compare discretization schemes for stochastic integrals of the form

$$\int_0^T X_{t-} dY_t,$$

where X_t and Y_t are semimartingales with jumps, with the aim of identifying asymptotically optimal schemes. In particular we wish to understand the impact of small jumps of X on the discretization error, and therefore we assume that X has no continuous local martingale part, see the discussion in Section 2 (Remark 2). Motivated by the decomposition (6), we will measure the error associated to a discretized strategy X' by the L^2 criterion

$$E \left[\left(\int_0^T (X_{t-} - X'_{t-}) dY_t \right)^2 \right].$$

Also, to each discretized strategy X' we associate various cost functionals depending in particular on the readjustment dates, which may represent the effect of transaction costs induced by the strategy X' . For example, the cost functional can simply be given by the expected number of readjustment dates. In our framework, a discretization strategy will be said to be optimal for a given cost functional if no strategy has (asymptotically, for large costs) a lower discretization error and a smaller cost.

In this work, motivated by the representation (4) and the readjustment rules used by market practitioners, we focus on discretization strategies based on the exit times of X out of random intervals and characterize explicitly the asymptotic behavior of the associated errors and costs. This allows us to determine the optimal intervals. In particular, we show that in the case where the cost functional is simply the expected number of discretization dates, the error associated to our optimal strategy with the cost equal to N , converges to zero as $N \rightarrow \infty$

at a faster rate than the error obtained by readjusting at N equally spaced dates.

In the case of exponential Lévy models, we obtain an explicit representation for the optimal discretization dates, which is similar to (4), but includes two “tuning” parameters: an index which determines the effect of transaction costs (fixed, proportional, etc.) and the Blumenthal-Gettoor index measuring the activity of small jumps.

This paper is structured as follows. In Section 2, we define the error and cost functionals and introduce the notion of asymptotic optimality based on limiting behavior of these functionals. The class of discretization rules based on hitting times of the process X is also introduced here. Section 3.1 contains the main results of this paper which characterize the limiting behavior of the error and the cost functionals, as well as the explicit examples of optimal discretization strategies. Sections 4 and 6 contain the proofs of the main results and Section 5 gathers some technical lemmas needed in Section 6.

2 Framework

Asymptotic comparison of discretization rules We are interested in comparing different discretization rules for the stochastic integral

$$\int_0^T X_{t-} dY_t,$$

where X and Y are semimartingales, in terms of their limiting behavior when the number of discretization points tends to infinity. A *discretization rule* is a family of stopping times $(T_i^\varepsilon)_{i \geq 0}^{\varepsilon > 0}$ parameterized by a nonnegative integer i and a positive real ε , such that for every $\varepsilon > 0$, $0 = T_0^\varepsilon < T_1^\varepsilon < T_2^\varepsilon < \dots$, and $\sup\{i : T_i^\varepsilon \leq T\} < \infty$. For a fixed discretization rule and a fixed ε , we denote $\eta(t) = \sup\{T_i^\varepsilon : T_i^\varepsilon \leq t\}$ and $N_T = \sup\{i : T_i^\varepsilon \leq T\}$.

The performance of a discretization rule is measured by the error functional $\mathcal{E}(\varepsilon) : (0, \infty) \rightarrow [0, \infty)$ and the cost functional $\mathcal{C}(\varepsilon) : (0, \infty) \rightarrow [0, \infty)$. We assume that the cost functional is such that

$$\lim_{\varepsilon \downarrow 0} \mathcal{C}(\varepsilon) = +\infty.$$

For $C > 0$ sufficiently large, we define

$$\varepsilon(C) = \inf\{\varepsilon > 0 : \mathcal{C}(\varepsilon) < C\}$$

and $\bar{\mathcal{E}}(C) := \mathcal{E}(\varepsilon(C))$.

Definition 1. We say that the discretization rule A asymptotically dominates the rule B if

$$\limsup_{C \rightarrow \infty} \frac{\bar{\mathcal{E}}^A(C)}{\bar{\mathcal{E}}^B(C)} \leq 1.$$

In order to apply Definition 1, the following lemma will be very useful.

Lemma 1. *Assume that the cost and error functionals are such that there exist $a > 0$ and $b > 0$ such that for two discretization rules A and B ,*

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \varepsilon^{-a} \mathcal{E}^A(\varepsilon) &= \hat{\mathcal{E}}^A \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \varepsilon^b \mathcal{C}^A(\varepsilon) = \hat{\mathcal{C}}^A, \\ \lim_{\varepsilon \downarrow 0} \varepsilon^{-a} \mathcal{E}^B(\varepsilon) &= \hat{\mathcal{E}}^B \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \varepsilon^b \mathcal{C}^B(\varepsilon) = \hat{\mathcal{C}}^B. \end{aligned} \quad (7)$$

for some constants $\hat{\mathcal{E}}^A$, $\hat{\mathcal{C}}^A$, $\hat{\mathcal{E}}^B$ and $\hat{\mathcal{C}}^B$. Then

$$\limsup_{C \rightarrow \infty} \frac{\bar{\mathcal{E}}^A(C)}{\bar{\mathcal{E}}^B(C)} = \frac{\hat{\mathcal{E}}^A}{\hat{\mathcal{E}}^B} \left(\frac{\hat{\mathcal{C}}^A}{\hat{\mathcal{C}}^B} \right)^{\frac{a}{b}}.$$

Proof. We remark that under the assumptions of the lemma,

$$\varepsilon(C) \sim \left(\frac{C}{\hat{\mathcal{C}}} \right)^{-\frac{1}{b}}, \quad C \rightarrow \infty,$$

and the rest of the proof follows easily. \square

Definition of the error and cost functionals We consider the error functional given by the L^2 error

$$\mathcal{E}(\varepsilon) := E \left[\left(\int_0^T (X_{t-} - X_{\eta(t)-}) dY_t \right)^2 \right] \quad (8)$$

and a family of cost functionals of the form

$$\mathcal{C}^\beta(\varepsilon) = E \left[\sum_{i \geq 1: T_i^\varepsilon \leq T} |X_{T_i^\varepsilon} - X_{T_{i-1}^\varepsilon}|^\beta \right]. \quad (9)$$

The case $\beta = 0$ correspond to a fixed cost for each discretization point, and the case $\beta = 1$ corresponds to proportional costs. The index β will be omitted whenever this does not lead to confusion.

Discretization rules based on hitting times We shall consider discretizations based on the hitting times of the process X . Such a discretization rule is defined by a pair of positive \mathbb{F} -adapted càdlàg processes $(\bar{a}_t)_{t \geq 0}$ and $(\underline{a}_t)_{t \geq 0}$. The discretization dates are then given by

$$T_{i+1}^\varepsilon = \inf\{t > T_i^\varepsilon : X_t \notin (X_{T_i^\varepsilon} - \varepsilon \underline{a}_{T_i^\varepsilon}, X_{T_i^\varepsilon} + \varepsilon \bar{a}_{T_i^\varepsilon})\}.$$

Remark 1. Consider the discretization rules $A = (\underline{a}, \bar{a})$ and $B = (k\underline{a}, k\bar{a})$ with $k > 0$. These two strategies satisfy $\bar{\mathcal{E}}^A(C) = \bar{\mathcal{E}}^B(C)$ for all $C > 0$. Therefore, the optimal strategies will be determined up to a multiplicative constant.

Assumptions Our first main result describing the behavior of the error functional will be obtained under the assumptions (H_Y) , (H_X) and (H_{loc}^1) stated below.

(H_Y) We assume that the process Y is a \mathbb{F} -local martingale, whose predictable quadratic variation satisfies $\langle Y \rangle_t = \int_0^t A_s ds$, where the process (A_t) is assumed càdlàg, locally bounded, and satisfies

$$E \left[\sup_{0 \leq s \leq T} \max(\underline{a}_s, \bar{a}_s)^2 \int_0^T A_t dt \right] < \infty.$$

Under this assumption the error functional (8) becomes

$$\mathcal{E}(\varepsilon) = E \left[\int_0^T (X_t - X_{\eta(t)})^2 A_t dt \right],$$

(H_X) The process X is a semimartingale defined via the stochastic representation

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \int_{|z| \leq 1} z(M - \mu)(ds \times dz) + \int_0^t \int_{|z| > 1} zM(ds \times dz), \quad (10)$$

where M is the jump measure of X and μ is its predictable compensator, which satisfies additionally $\mu(dt \times dz) = dt \times \lambda_t K_t(z) \nu(dz)$, where λ is a positive càdlàg process and ν is a Lévy measure satisfying

$$x^\alpha \nu((x, \infty)) \rightarrow c_+ \quad \text{and} \quad x^\alpha \nu((-\infty, -x)) \rightarrow c_- \quad \text{when} \quad x \rightarrow 0. \quad (H_\alpha)$$

for some $\alpha \in (1, 2)$ and constants $c_+ \geq 0$ and $c_- \geq 0$ with $c_+ + c_- > 0$.

(H_{loc}^ρ) There exists an increasing sequence of stopping times (τ_n) with $\tau_n \rightarrow T$ such that for every n ,

$$\int_0^{\tau_n} \int_{\mathbb{R}} |\sqrt{K_t(z)} - 1|^{2\rho} \nu(dz) dt < K_n,$$

$$\frac{1}{K_n} \leq \lambda_t, \underline{a}_t, \bar{a}_t \leq K_n \quad \text{and} \quad |b_t| \leq K_n \quad \text{for} \quad 0 \leq t \leq \tau_n \quad \text{and} \quad \text{some constant } K_n > 0.$$

Remark 2. — In this paper, we focus on semimartingales for which the local martingale part is purely discontinuous, with the aim of determining the effect of small jumps on the convergence rate of the discretization error. Therefore, we do not include a continuous local martingale part in the dynamics of X . The dynamics of Y can, in principle, include such a continuous local martingale part, however in the usual financial models, when X has no continuous local martingale part, this is also the case for Y .

– The parameter α measures the activity of small jumps of the process X . In the case where X is a Lévy process (that is, b , λ and K are deterministic and time-independent), the parameter α coincides with the so called Blumenthal-Gettoor index of X (see [3]).

– Finally, the assumption $1 < \alpha < 2$ implies that X has infinite variation and ensures that the local behavior of the process is determined by the jumps rather than by the drift part (see [17]). Note that in a recent statistical study on liquid assets [1], the jump activity index defined similarly to our parameter α was estimated between 1.4 and 1.7.

To obtain our second main result concerning the behavior of the cost functional, we shall need the following additional technical assumptions.

(C₁) There exists $C < \infty$ such that

$$\int_{|z|>x} K_t(z) \nu(dz) < Cx^{-\alpha}, \quad \forall x > 0.$$

(C₂) For some $\delta \in (0, 1)$ with $\beta(1 + \delta) < \alpha$,

$$\begin{aligned} & E \left[\sup_{0 \leq s \leq T} (\max\{\underline{a}_s^{\beta-1}, \bar{a}_s^{\beta-1}\}^{1+\delta} + \max\{\underline{a}_s^{(1+\delta)\beta-1}, \bar{a}_s^{(1+\delta)\beta-1}\}) \int_0^T |b_s|^{1+\delta} ds \right] \\ & + E \left[\sup_{0 \leq s \leq T} \max\{\underline{a}_s, \bar{a}_s\}^{(\beta \vee (2-\alpha))(1+\delta)} \min\{\underline{a}_s, \bar{a}_s\}^{((\beta-2) \wedge (-\alpha))(1+\delta)} \int_0^T \lambda_s^{1+\delta} ds \right] < \infty. \end{aligned}$$

(C'₂) For some $\delta \in (0, 1)$,

$$E \left[\sup_{0 \leq s \leq T} \min(\underline{a}_s, \bar{a}_s)^{-\alpha(1+\delta)} \int_0^T \lambda_t^{1+\delta} dt + \sup_{0 \leq s \leq T} \min(\underline{a}_s, \bar{a}_s)^{-1-\delta} \int_0^T |b_t|^{1+\delta} dt \right] < \infty.$$

3 Main results

In this section, we first characterize the asymptotic behavior of the error and cost functionals for small ε . From these results we then derive the asymptotically optimal discretization strategies using Lemma 1.

3.1 Asymptotic behavior of the error and cost functionals

Theorem 1. *Under the assumptions (H_Y) , (H_X) and (H_{loc}^1) ,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathcal{E}(\varepsilon) = E \left[\int_0^T A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] \quad (11)$$

where, for $\underline{a}, \bar{a} \in (0, \infty)$,

$$f(\underline{a}, \bar{a}) = E \left[\int_0^{\tau^*} (X_t^*)^2 dt \right], \quad g(\underline{a}, \bar{a}) = E[\tau^*]$$

with $\tau^* = \inf\{t \geq 0 : X_t^* \notin (-\underline{a}, \bar{a})\}$, where X^* is a strictly α -stable process with Lévy density

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}.$$

Theorem 2. *We use the notation of Theorem 1.*

i. *Let the assumptions (H_Y) , (H_X) , (H_{loc}^1) , (C_1) and (C_2') be satisfied. Then,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\alpha \mathcal{C}^0(\varepsilon) = E \left[\int_0^T \frac{\lambda_t}{g(\underline{a}_t, \bar{a}_t)} dt \right]. \quad (12)$$

ii. *Let the cost function be of the form (9) with $\beta \in (0, \alpha)$ and assume that (H_Y) , (H_X) , (H_{loc}^ρ) , (C_1) and (C_2) hold true for some $\rho > \frac{\alpha}{\alpha-\beta} \vee 2$. Then,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\alpha-\beta} \mathcal{C}(\varepsilon) = E \left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right], \quad (13)$$

where

$$u^\beta(\underline{a}, \bar{a}) = E[|X_{\tau^*}^*|^\beta] < \infty.$$

3.2 Application: computing the optimal barriers

The above results will now be used to compute the asymptotically optimal barriers \underline{a}_t and \bar{a}_t . In view of Lemma 1, we need to solve the optimization problem

$$\min_{\underline{a}, \bar{a}} E \left[\int_0^T A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] E \left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right]^{\frac{2}{\alpha-\beta}}.$$

Since the functions f , u and g are homogeneous (by the scaling property), the solution is only determined up to multiplying \underline{a} and \bar{a} by the same constant, and therefore, the set of solutions to the above problem coincides with the set of solutions of

$$\min_{\underline{a}, \bar{a}} E \left[\int_0^T A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] \quad \text{subject to} \quad E \left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] = C,$$

for $C \in (0, \infty)$. By the method of Lagrange multipliers we then conclude that \underline{a} and \bar{a} are solutions to

$$(\underline{a}_t, \bar{a}_t) = \arg \min \left\{ A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} + c \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} \right\},$$

provided that the resulting \underline{a}_t and \bar{a}_t satisfy the necessary assumptions. Here, c (the Lagrange multiplier) is a constant chosen by the trader depending on the maximum acceptable cost: the bigger c , the smaller will be the cost of the strategy and, consequently the bigger its error. The functions f , g and u appearing above must in general be computed numerically, however, in the case when the limiting process X^* is a symmetric stable process, which corresponds for example to the CGMY model very popular in practice [6], the results are completely explicit, as will be shown in the next paragraph.

Case of locally symmetric Lévy measures Assume that the limiting process X^* is a symmetric stable process with characteristic function $E[e^{iuX_t^*}] = e^{t\sigma|u|^\alpha}$ (this means that (H_α) is satisfied with $c_+ = c_-$), and that $\beta \leq 1$. To simplify notation, we write $a_t := \frac{\underline{a}_t + \bar{a}_t}{2}$ and $\theta_t = \frac{\bar{a}_t - \underline{a}_t}{\bar{a}_t + \underline{a}_t}$. Then, using the results from the Appendix, we compute:

$$\begin{aligned} \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} &= \frac{\alpha}{(\alpha+2)(\alpha+1)} a_t^2 (1 + \theta_t^2(1+\alpha)) \\ \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} &= \frac{\sigma\Gamma(1+\alpha) \sin \frac{\pi\alpha}{2}}{\pi} \\ &\quad \times \int_0^\infty z^{-\alpha/2} (z + 2a_t)^{-\alpha/2} (|z + a_t(1+\theta_t)|^{\beta-1} + |z + a_t(1-\theta_t)|^{\beta-1}) dz. \end{aligned}$$

For fixed a_t , both ratios are minimal when $\theta = 0$ (for the second functional this follows from the convexity of the function $x \mapsto x^{\beta-1}$ for $x \geq 0$ and $\beta \leq 1$). Therefore, it is optimal to choose $\theta_t \equiv 0$ or $\underline{a}_t \equiv \bar{a}_t$. Therefore, the optimal process (a_t) minimizes

$$E \left[\int_0^T A_t a_t^2 dt \right] E \left[\int_0^T \lambda_t a_t^{\beta-\alpha} dt \right]^{\frac{2}{\alpha-\beta}}.$$

As before, this can be easily solved using the method of Lagrange multipliers and we find that the optimal interval size is

$$a_t = c \left(\frac{\lambda_t}{A_t} \right)^{\frac{1}{2+\alpha-\beta}},$$

where c is a constant which is chosen by the trader depending on the maximum acceptable cost.

Let us now specialize to the case $\beta = 0$ (which means that the cost is equal to the expected number of rebalancings). By the same logic as in Lemma 1, we get that for the optimal solution, the error functional for fixed cost, $\bar{\mathcal{E}}(C)$, satisfies

$$\bar{\mathcal{E}}(C) \sim C^{-\frac{2}{\alpha}} \left(E \left[\int_0^T A_t^{\frac{\alpha}{2+\alpha}} \lambda_t^{\frac{2}{2+\alpha}} dt \right] \right)^{\frac{2+\alpha}{\alpha}}, \quad C \rightarrow \infty.$$

On the other hand, for equidistant rebalancing dates, it is known that under sufficient regularity, the L^2 discretization error of the quadratic hedging strategy in exponential Lévy models is *inversely proportional* to the number of rebalancings (see [5]). This means that while in diffusion models, asymptotically optimal hedging reduces the error without modifying the rate at which the error decreases with the number of rebalancing, in pure jump models, the asymptotically optimal discretization also improves the rate of convergence.

Exponential Lévy models To make the formulae even more explicit, assume that Y is an exponential of a Lévy process: $Y_t = e^{Z_t}$ where Z is a Lévy process without diffusion part, and whose Lévy measure has a density $\nu(x)$ satisfying $\nu(x) \sim \frac{c}{|x|^{1+\alpha}}$, $x \rightarrow 0$. Then $A_t = \Sigma Y_t^2$ with $\Sigma = \int (e^z - 1)^2 \nu(z) dz$. The quadratic hedging strategy in this case has been given by several authors [16, 8] and is known to have a Markov structure: $X_t = \phi(t, Y_t)$ for a deterministic function ϕ . The compensator μ of the jump measure of X is then equal to dt times the image measure of ν by the mapping $\delta(z) : z \mapsto \phi(t, Y_t e^z) - \phi(t, Y_t)$. Assume furthermore that the function ϕ is sufficiently regular, which is often the case in practice, and that δ' is strictly positive. Then,

$$\mu(dt \times dz) = dt \times \nu(\delta^{-1}(z)) \frac{dy}{\delta'(\delta^{-1}(z))}$$

In this case,

$$\lambda_t = \lim_{z \rightarrow 0} \frac{\nu(\delta^{-1}(z))}{\nu(z) \delta'(\delta^{-1}(z))} = (\delta'(0))^\alpha = \left(Y_t \frac{\partial \phi(t, Y_t)}{\partial Y} \right)^\alpha$$

and therefore

$$a_t = c \left(\frac{\partial \phi(t, Y_t)}{\partial Y} \right)^{\frac{\alpha}{2+\alpha-\beta}} Y_t^{\frac{\alpha-2}{\alpha-\beta+2}}.$$

Finally, when $\beta = 0$ and $\alpha \rightarrow 2$, we find that the optimal size of the rebalancing interval is proportional to the square root of $\frac{\partial \phi(t, Y_t)}{\partial Y}$ (the gamma), which is consistent with the results of Fukasawa [11], quoted in the introduction.

4 Proof of Theorem 1

Step 1. Reduction to the case of bounded coefficients. In the proofs of Theorems 1 and 2, we will replace the assumption (H_{loc}^ρ) with the following stronger one:

(H'_ρ) There exists a constant $K > 0$ such that $\frac{1}{K} \leq \lambda_t, \underline{a}_t, \bar{a}_t \leq K$, $|A_t| \leq K$ and $|b_t| \leq K$ for $0 \leq t \leq T$ and moreover the process (Z_t) defined by

$$Z_t = \mathcal{E} \left(\int_0^\cdot (K_s^{-1}(z) - 1)(M - \mu)(ds \times dz) \right)_t,$$

is a martingale and satisfies

$$E^Q \left[\sup_{0 \leq t \leq T} |Z_t|^{-\rho} \right] < \infty \quad \text{and} \quad E \left[\sup_{0 \leq t \leq T} Z_t \right] < \infty,$$

where Q is the probability measure defined by

$$\frac{dQ}{dP}|_{\mathcal{F}_T} := Z_T.$$

Indeed, we have the following lemma.

Lemma 2. *Assume that (11) holds under the assumptions (H_Y) , (H_X) and (H'_1) . Then Theorem 1 holds.*

Proof. First, observe that for every n ,

$$\begin{aligned} & E \left[\left\{ \int_0^{\tau_n} \int_{\mathbb{R}} (K_s^{-1}(z) - 1)^2 M(ds \times dz) \right\}^{\frac{1}{2}} \right] \\ & \leq E \left[\left\{ \int_0^{\tau_n} \int_{|K_s^{-1}(z) - 1| \leq \frac{1}{2}} (K_s^{-1}(z) - 1)^2 M(ds \times dz) \right\}^{\frac{1}{2}} \right] \\ & \quad + E \left[\left\{ \int_0^{\tau_n} \int_{|K_s^{-1}(z) - 1| > \frac{1}{2}} (K_s^{-1}(z) - 1)^2 M(ds \times dz) \right\}^{\frac{1}{2}} \right], \end{aligned}$$

which is finite by Proposition II.1.28 in [15] since by assumption (H_{loc}^1) ,

$$\begin{aligned} & E \left[\int_0^{\tau_n} \int_{|K_s^{-1}(z) - 1| \leq \frac{1}{2}} (K_s^{-1}(z) - 1)^2 \mu(ds \times dz) \right]^{\frac{1}{2}} \\ & \quad + E \left[\int_0^{\tau_n} \int_{|K_s^{-1}(z) - 1| > \frac{1}{2}} |K_s^{-1}(z) - 1| \mu(ds \times dz) \right] < \infty. \end{aligned}$$

This implies that the process

$$L_t = \int_0^t \int_{\mathbb{R}} (K_s^{-1}(z) - 1)(M - \mu)(ds \times dz)$$

is a local martingale and satisfies $E[[L]_{T \wedge \tau_n}^{\frac{1}{2}}] < \infty$ for every n (see Definition II.1.27 in [15]). The process $Z_t := \mathcal{E}(L)_t$ is then also well defined and we take $\sigma_n := \tau_n \wedge \inf\{t : Z_t \geq n\}$. Then,

$$\begin{aligned} \sup_{0 \leq t \leq T} Z_{t \wedge \sigma_n} & \leq n + |\Delta Z_{\sigma_n}| 1_{\sigma_n \leq T} \leq n + [Z]_{\sigma_n \wedge T}^{\frac{1}{2}} = n + \left(\int_0^{\sigma_n \wedge T} Z_{t-}^2 d[L]_t \right)^{\frac{1}{2}} \\ & \leq n + n[L]_{\sigma_n \wedge T}^{\frac{1}{2}}, \end{aligned}$$

which is integrable. Therefore, we can define a new probability measure Q^n via

$$\frac{dQ^n}{dP}|_{\mathcal{F}_t} = Z_{t \wedge \sigma_n}.$$

By Girsanov theorem (Theorem III.5.24 in [15]), M is a random measure with predictable compensator $\mu^{Q^n} := dt \times \lambda_t \nu(dz)$ under Q^n on $\{t \leq \sigma_n\}$ and

$$Z_{t \wedge \sigma_n}^{-1} = \mathcal{E} \left(\int_0^\cdot (K_s(z) - 1)(M - \mu^{Q^n})(ds \times dz) \right)_{t \wedge \sigma_n}.$$

Therefore, by similar arguments to above, we can find an increasing sequence of stopping times (γ_n) with $\gamma_n \rightarrow T$ and such that both

$$E \left[\sup_{0 \leq t \leq T} Z_{t \wedge \gamma_n} \right] < \infty \quad \text{and} \quad E^{Q^n} \left[\sup_{0 \leq t \leq T} Z_{t \wedge \gamma_n}^{-1} \right] < \infty,$$

Now we define $Y_t^n = Y_{t \wedge \gamma_n}$ and X^n via Equation (10) replacing the coefficients λ_t, b_t and $K_t(z)$ with $\lambda_t^n := \lambda_{t \wedge \gamma_n}, b_t^n := b_{t \wedge \gamma_n}$ and $K_t^n(z) = K_t(z)1_{t \leq \gamma_n} + 1_{t > \gamma_n}$. Moreover, we define $\underline{a}_t^n := \underline{a}_{t \wedge \gamma_n}, \bar{a}_t^n := \bar{a}_{t \wedge \gamma_n}$. The stopping times $T_i^{\varepsilon, n}$ and $\eta^n(t)$ are defined similarly. Remark that $A_t^n := A_t 1_{t \leq \gamma_n}$ satisfies $\int_0^t A_s^n ds = \langle Y^n \rangle_t$, that X^n coincides with X on the interval $[0, \gamma_n]$ and that the new coefficients satisfy the assumption (H'_1) for K sufficiently large. Consequently,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \varepsilon^{-2} E \left[\int_0^{\gamma_n} (X_t - X_{\eta^n(t)})^2 A_t dt \right] &= \lim_{\varepsilon \downarrow 0} \varepsilon^{-2} E \left[\left(\int_0^T (X_t^n - X_{\eta^n(t)}^n)^2 dY_t^n \right)^2 \right] \\ &= E \left[\int_0^T A_t^n \frac{f(\underline{a}_t^n, \bar{a}_t^n)}{g(\underline{a}_t^n, \bar{a}_t^n)} dt \right] = E \left[\int_0^{\gamma_n} A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right], \end{aligned}$$

which implies, by Assumption (H_Y) , that

$$E \left[\int_0^{\gamma_n} A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] \leq E \left[\sup_{0 \leq s \leq T} \max(\underline{a}_s, \bar{a}_s)^2 \int_0^T A_t dt \right],$$

and so by Fatou's lemma,

$$E \left[\int_0^T A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] \leq E \left[\sup_{0 \leq s \leq T} \max(\underline{a}_s, \bar{a}_s)^2 \int_0^T A_t dt \right].$$

Therefore, by dominated convergence

$$\lim_n E \left[\int_{\gamma_n}^T A_t \frac{f(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] = 0.$$

On the other hand,

$$\varepsilon^{-2} E \int_{\gamma_n}^T (X_t - X_{\eta^n(t)})^2 A_t dt \leq E \left[\sup_{0 \leq s \leq T} \max(\underline{a}_s, \bar{a}_s)^2 \int_{\gamma_n}^T A_t dt \right].$$

The right-hand side does not depend on ε and converges to zero as $n \rightarrow \infty$ by the dominated convergence theorem. Therefore, the left-hand side can be made arbitrarily small independently of ε , and the result follows. \square

Step 2. Change of probability measure. We first prove the following important lemma.

Lemma 3. *Under the assumption H'_1*

$$\lim_{\varepsilon \rightarrow 0} \sup_{i: T_i^\varepsilon \leq T} (T_{i+1}^\varepsilon - T_i^\varepsilon) = 0$$

almost surely.

Proof. In this proof, let us fix $\omega \in \Omega$. By way of contradiction, assume that there exists a constant $C > 0$ and a sequence $\{\varepsilon_n\}_{n \geq 0}$ converging to zero such that for every n , there exists $i(n)$ with $T_{i(n)+1}^{\varepsilon_n} - T_{i(n)}^{\varepsilon_n} > C$. From the sequences $\{T_{i(n)+1}^{\varepsilon_n}\}_n$ and $\{T_{i(n)}^{\varepsilon_n}\}_n$ we can extract two subsequences $\{T_{i(\phi(n))+1}^{\varepsilon_{\phi(n)}}\}_n$ and $\{T_{i(\phi(n))}^{\varepsilon_{\phi(n)}}\}_n$ converging to some limiting values $T_1 < T_2$. For n big enough, there exists a nonempty interval \mathcal{I} which is a subset of both (T_1, T_2) and $(T_{i(\phi(n))+1}^{\varepsilon_{\phi(n)}}, T_{i(\phi(n))}^{\varepsilon_{\phi(n)}})$. Now using that $\sup_{t, s \in (T_{i(\phi(n))+1}^{\varepsilon_{\phi(n)}}, T_{i(\phi(n))}^{\varepsilon_{\phi(n)}})} |X_t - X_s| \leq c\varepsilon_{\phi(n)}$, we obtain that $\sup_{s, t \in \mathcal{I}} |X_t - X_s| = 0$, which is cannot hold since X is an infinite activity process. \square

Let $\Delta T_{i+1} = T_{i+1} \wedge T - T_i \wedge T$. The goal of this step is to show that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \varepsilon^{-2} E \left[\int_0^T (X_t - X_{\eta(t)})^2 A_t dt \right] \\ = \lim_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=1}^{\infty} Z_{T_i \wedge T}^{-1} A_{T_i \wedge T} \varepsilon^{-2} \int_{T_i \wedge T}^{T_{i+1} \wedge T} (X_t - X_{T_i})^2 dt \right]. \end{aligned} \quad (14)$$

We have

$$\begin{aligned} \varepsilon^{-2} E \left[\int_0^T (X_t - X_{\eta(t)})^2 A_t dt \right] \\ = \varepsilon^{-2} \sum_{i=0}^{+\infty} E \left[\int_{T_i \wedge T}^{T_{i+1} \wedge T} (X_t - X_{T_i})^2 (A_t - A_{T_i}) dt \right] \\ + \varepsilon^{-2} \sum_{i=0}^{+\infty} E^Q \left[Z_{T_{i+1} \wedge T}^{-1} A_{T_i} \int_{T_i \wedge T}^{T_{i+1} \wedge T} (X_t - X_{T_i})^2 dt \right]. \end{aligned}$$

Since for $t \in [T_i, T_{i+1})$, $(X_t - X_{T_i})^2 \leq \varepsilon^2$, using the boundedness of A , (14) will follow provided we show that

$$\lim_{\varepsilon \downarrow 0} \sum_{i=0}^{+\infty} E \left[\int_{T_i \wedge T}^{T_{i+1} \wedge T} |A_t - A_{T_i}| dt \right] = 0 \quad (15)$$

and

$$\lim_{\varepsilon \downarrow 0} \sum_{i=0}^{+\infty} E^Q \left[|Z_{T_{i+1} \wedge T}^{-1} - Z_{T_i \wedge T}^{-1}| \Delta T_{i+1} \right] = 0. \quad (16)$$

The limit (15) follows from the dominated convergence theorem (A is bounded by assumption (H'_1) and $A_{\eta(t)} \rightarrow A_t$ a.e. on $[0, T]$ since A is càdlàg and by Lemma 3). Using the fact that Z^{-1} has finite quadratic variation together with Lemma 3 and Cauchy Schwarz inequality, we get that, in probability,

$$\lim_{\varepsilon \downarrow 0} \sum_{i=0}^{+\infty} |Z_{T_{i+1} \wedge T}^{-1} - Z_{T_i \wedge T}^{-1}| \Delta T_{i+1} = 0.$$

Then (16) follows from the integrability of $\sup_{t \in [0, T]} Z_t^{-1}$, which is part of Assumption (H'_1) .

Step 3. First, observe that by the dominated convergence theorem, since $\sup_i \Delta T_i$ tends to zero, (14) is equal to

$$S_1 := \lim_{\varepsilon \downarrow 0} S_1^\varepsilon \quad \text{with} \quad S_1^\varepsilon := E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_{T_i}^{T_{i+1}} (X_t - X_{T_i})^2 dt \right] \right].$$

For this expression to be well defined we extend the processes λ , b , \underline{a} , \bar{a} by arbitrary constant values beyond T and define the process X for $t \geq T$ accordingly.

Define a family of continuous increasing processes $(\Lambda_s(t))_{t \geq 0}$ indexed by $s \geq 0$ by $\Lambda_s(t) = \int_s^{s+t} \lambda_r dr$, the family of filtrations $\mathcal{G}_t^i = \mathcal{F}_{T_i+t}$ and of processes $(\tilde{X}_t^i)_{t \geq 0}$ and $(\hat{X}_t^i)_{t \geq 0}$ by

$$\hat{X}_t^i = X_{T_i + \Lambda_{T_i}^{-1}(t)} - X_{T_i} - \int_{T_i}^{T_i + \Lambda_{T_i}^{-1}(t)} b_s ds, \quad \tilde{X}_t^i = X_{T_i + \Lambda_{T_i}^{-1}(t)} - X_{T_i}.$$

The process $(\hat{X}_t^i)_{t \geq 0}$ is a (\mathcal{G}_t^i) -semimartingale with (deterministic) characteristics $(0, \nu, 0)$ under Q , therefore, it is a (\mathcal{G}_t^i) -Lévy process under Q (Theorem II.4.15 in [15]).

Let $\tilde{\tau}_i = \inf\{t \geq 0 : \tilde{X}_t^i \notin [-\underline{a}_{T_i} \varepsilon, \bar{a}_{T_i} \varepsilon]\}$. Using a change of variable formula we obtain that

$$\int_{T_i}^{T_{i+1}} (X_t - X_{T_i})^2 dt = \int_0^{\tilde{\tau}_i} \frac{\tilde{X}_s^2}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))} ds.$$

Using the càdlàg property of λ together with the various boundedness assumptions and the integrability of $\sup_{0 \leq t \leq T} Z_t^{-1}$, we easily get that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} \frac{A_{T_i} Z_{T_i}^{-1}}{\lambda_{T_i}} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} \tilde{X}_t^2 dt \right] \right].$$

Then we obviously have that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} \frac{A_{T_i} Z_{T_i}^{-1}}{\lambda_{T_i}} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q [T_{i+1} - T_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} \tilde{X}_t^2 dt \right] \right].$$

Now remark that

$$T_{i+1} - T_i = \int_0^{\tilde{\tau}_i} \frac{ds}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))}. \quad (17)$$

Then,

$$\begin{aligned} & E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} \frac{A_{T_i} Z_{T_i}^{-1}}{\lambda_{T_i}} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q [T_{i+1} - T_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} \tilde{X}_t^2 dt \right] \right] \\ &= E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q [\tilde{\tau}_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} \tilde{X}_t^2 dt \right] \right] + R^\varepsilon \end{aligned}$$

with

$$|R^\varepsilon| \leq CE^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} Z_{T_i}^{-1} (T_{i+1} - T_i) \left| \frac{\lambda_{T_i}^{-1} E_{\mathcal{F}_{T_i}} [\tilde{\tau}_i] - E_{\mathcal{F}_{T_i}} \left[\int_0^{\tilde{\tau}_i} \frac{ds}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))} \right]}{E_{\mathcal{F}_{T_i}} \left[\int_0^{\tilde{\tau}_i} \frac{ds}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))} \right]} \right| \right].$$

Using (17), we obtain that

$$\begin{aligned} |R^\varepsilon| &\leq CE^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} Z_{T_i}^{-1} \left| \lambda_{T_i}^{-1} E_{\mathcal{F}_{T_i}} [\tilde{\tau}_i] - E_{\mathcal{F}_{T_i}} \left[\int_0^{\tilde{\tau}_i} \frac{ds}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))} \right] \right| \right] \\ &\leq CE^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} Z_{T_i}^{-1} \int_0^{\tilde{\tau}_i} \left| \frac{1}{\lambda_{T_i}} - \frac{1}{\lambda(T_i + \Lambda_{T_i}^{-1}(s))} \right| ds \right] \\ &\leq CE^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} Z_{T_i}^{-1} \int_{T_i}^{T_{i+1}} \left| \frac{1}{\lambda_{T_i}} - \frac{1}{\lambda(s)} \right| ds \right], \end{aligned}$$

which is easily shown to converge to zero. Consequently, we conclude that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q [\tilde{\tau}_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} \tilde{X}_t^2 dt \right] \right]. \quad (18)$$

Step 4. Comparison of hitting times and associated integrals. We start with the following lemma

Lemma 4. *Let $\kappa \in \mathbb{R}_+$ and $n \in \mathbb{N}$. Then,*

$$\underline{f}_\varepsilon^{\kappa, n}(\underline{a}_{T_i}, \bar{a}_{T_i}) \leq E_{\mathcal{F}_{T_i}} \left[\left(\int_0^{\tilde{\tau}_i} |\hat{X}_t|^\kappa dt \right)^n \right] \leq \bar{f}_\varepsilon^{\kappa, n}(\underline{a}_{T_i}, \bar{a}_{T_i})$$

whenever the expression in the middle is well defined, where $\underline{f}_\varepsilon$ and \bar{f}_ε are deterministic functions defined by

$$\begin{aligned} \underline{f}_\varepsilon^{\kappa, n}(a, b) &= E \left[\left(\int_0^{\hat{\tau}_1} |\hat{X}_t|^\kappa dt \right)^n \right] \\ \bar{f}_\varepsilon^{\kappa, n}(a, b) &= E \left[\left(\int_0^{\hat{\tau}_2 \wedge \hat{\tau}^j} |\hat{X}_t|^\kappa dt \right)^n \right] \end{aligned}$$

with

$$\begin{aligned}\hat{\tau}_1 &= \inf\{t : \hat{X}_t \leq -a\varepsilon + tK^2 \quad \text{or} \quad \hat{X}_t \geq b\varepsilon - tK^2\} \\ \hat{\tau}_2 &= \inf\{t : \hat{X}_t \leq -a\varepsilon - tK^2 \quad \text{or} \quad \hat{X}_t \geq b\varepsilon + tK^2\} \\ \hat{\tau}^j &= \inf\{t : |\Delta \hat{X}_t| \geq \varepsilon(a+b)\}.\end{aligned}$$

The proof follows from the fact that $|\tilde{X}_t - \hat{X}_t| \leq tK^2$ and that \hat{X} is a \mathcal{G}_t^i -Lévy process under Q , and that a jump of size greater than $\varepsilon(a+b)$ immediately takes the process \tilde{X} out of the interval.

Lemma 5.

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-(\kappa+\alpha)n} \underline{f}_\varepsilon^{\kappa,n}(a,b) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-(\kappa+\alpha)n} \overline{f}_\varepsilon^{\kappa,n}(a,b) = f^{*,\kappa,n}(a,b) \quad (19)$$

uniformly on $(a,b) \in [a_1, a_2] \times [b_1, b_2]$ for all $0 < a_1 \leq a_2 < \infty$ and $0 < b_1 \leq b_2 < \infty$, with

$$f^{*,\kappa,n}(a,b) = E \left[\left(\int_0^{\tau^*} |X_t^*|^\kappa dt \right)^n \right],$$

where X^* is a strictly α -stable process with Lévy density

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}.$$

and $\tau^* = \inf\{t \geq 0 : X_t^* \notin (-a,b)\}$.

Proof. Let us define $X_t^\varepsilon = \varepsilon^{-1} \hat{X}_{\varepsilon^\alpha t}$, $X_t^{\varepsilon,1} = X_t^\varepsilon - tK^2 \varepsilon^{\alpha-1}$, $X_t^{\varepsilon,2} = X_t^\varepsilon + tK^2 \varepsilon^{\alpha-1}$ and

$$\begin{aligned}\tau_1^{\varepsilon,1} &= \inf\{t, X_t^{\varepsilon,1} \leq -a\}, & \tau_1^{\varepsilon,2} &= \inf\{t, X_t^{\varepsilon,2} \geq b\}, \\ \tau_2^{\varepsilon,1} &= \inf\{t, X_t^{\varepsilon,2} \leq -a\}, & \tau_2^{\varepsilon,2} &= \inf\{t, X_t^{\varepsilon,1} \geq b\} \\ \tau_3^{\varepsilon,1} &= \inf\{t, X_t^\varepsilon \leq -a\}, & \tau_3^{\varepsilon,2} &= \inf\{t, X_t^\varepsilon \geq b\}.\end{aligned}$$

We write $\tau_i^\varepsilon = \tau_i^{\varepsilon,1} \wedge \tau_i^{\varepsilon,2}$ for $i = 1, 2, 3$. Similarly, we define $\tau^{j,\varepsilon} := \inf\{t : |\Delta X_t^\varepsilon| \geq (a+b)\}$. Observe that by a change of variable in the integral,

$$\begin{aligned}\varepsilon^{-(\kappa+\alpha)n} \underline{f}_\varepsilon^{\kappa,n}(a,b) &= E \left[\left(\int_0^{\tau_1^\varepsilon} |X_t^\varepsilon|^\kappa dt \right)^n \right], \\ \varepsilon^{-(\kappa+\alpha)n} \overline{f}_\varepsilon^{\kappa,n}(a,b) &= E \left[\left(\int_0^{\tau_2^\varepsilon \wedge \tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa dt \right)^n \right].\end{aligned}$$

From Lemma 11, we have that X_t^ε converges to X_t^* in Skorohod topology. From Skorohod representation theorem, there exists some probability space on which are defined a process Y^* and a family of processes Y^ε such that Y^ε and X^ε have the same law, Y^* and X^* have the same law and Y^ε converges to Y^* almost

surely, for the Skorohod topology.

This implies that $Y^{\varepsilon,1}$ and $Y^{\varepsilon,2}$ also converge to Y^* almost surely, where $Y_t^{\varepsilon,1} = Y_t^\varepsilon - tK^2\varepsilon^{\alpha-1}$ and $Y_t^{\varepsilon,2} = Y_t^\varepsilon + tK^2\varepsilon^{\alpha-1}$. Now using that the application which to a function f in the Skorohod space associates its first hitting time of a constant barrier is continuous at almost all f which are sample paths of strictly stable processes, see [17], we obtain that σ_i^ε converges almost surely to σ^* for $i = 1, 2, 3$, where σ_i^ε and σ^* are defined through $Y^{\varepsilon,1}$, $Y^{\varepsilon,2}$, Y^* in the same way as τ_i^ε and τ^* through $X^{\varepsilon,1}$, $X^{\varepsilon,2}$, X^* . Moreover, since $\sigma_3^\varepsilon \leq \sigma^{j,\varepsilon}$ for all ε , we also have that $\sigma_2^\varepsilon \wedge \sigma^{j,\varepsilon} \rightarrow \sigma^*$ almost surely.

Now remark that, almost surely, Y_t^ε converges almost everywhere in t to Y_t^* , see Proposition VI.2.3 in [15]. Therefore, since $|Y_t^\varepsilon|1_{t \leq \sigma_1^\varepsilon} \leq \max(a, b)$ and $|Y_t^\varepsilon|1_{t \leq \sigma^{j,\varepsilon} \wedge \sigma_2^\varepsilon} \leq \max(a, b) + K^2t$, using the dominated convergence theorem, we obtain that almost surely

$$\begin{aligned} \left(\int_0^{\sigma_1^\varepsilon} |Y_t^\varepsilon|^\kappa dt \right)^n &\rightarrow \left(\int_0^{\sigma^*} |Y_t^*|^\kappa dt \right)^n \\ \left(\int_0^{\sigma_2^\varepsilon \wedge \sigma^{j,\varepsilon}} |Y_t^\varepsilon|^\kappa dt \right)^n &\rightarrow \left(\int_0^{\sigma^*} |Y_t^*|^\kappa dt \right)^n. \end{aligned}$$

Finally, we deduce that

$$\begin{aligned} \left(\int_0^{\tau_1^\varepsilon} |X_t^\varepsilon|^\kappa dt \right)^n &\rightarrow \left(\int_0^{\tau^*} |X_t^*|^\kappa dt \right)^n, \\ \left(\int_0^{\tau_2^\varepsilon \wedge \tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa dt \right)^n &\rightarrow \left(\int_0^{\tau^*} |X_t^*|^\kappa dt \right)^n, \end{aligned}$$

in law.

Now remark that

$$P[\tau^{j,\varepsilon} > T] \leq \exp\{-T\varepsilon^\alpha \nu((-\infty, -(a+b)\varepsilon] \cup [(a+b)\varepsilon, \infty))\},$$

which, by our assumption (H_α) on the Lévy measure, implies that the family $(\tau^{j,\varepsilon})_{\varepsilon>0}$ has uniformly bounded exponential moment. This implies that the families

$$\left(\int_0^{\tau_2^\varepsilon \wedge \tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa dt \right)^n \quad \text{and} \quad \left(\int_0^{\tau_1^\varepsilon} |X_t^\varepsilon|^\kappa dt \right)^n = \left(\int_0^{\tau_1^\varepsilon \wedge \tau^{j,\varepsilon}} |X_t^\varepsilon|^\kappa dt \right)^n,$$

parameterized by ε , are uniformly integrable and therefore the proof of the convergence in (19) is complete.

It remains to show that the convergence in (19) is uniform in (a, b) over compact sets excluding zero. To do this, first observe that $f^{*,\kappa,n}(a, b)$ is continuous

in (a, b) on compact sets excluding zero (this is shown using essentially the same arguments as above: continuity of the exit times for stable processes plus uniform integrability). Secondly, since both $\underline{f}_\varepsilon^{\kappa, n}$ and $\overline{f}_\varepsilon^{\kappa, n}$ are increasing in a and b , a multidimensional version of Dini's theorem can be used to conclude that the convergence is indeed uniform. \square

Step 5. First, let us show that

$$S_1 = \lim_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q[\tilde{\tau}_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} \hat{X}_t^2 dt \right] \right].$$

Indeed, the absolute value of the difference between the expressions under the limit here and in (18) is bounded from above by

$$\begin{aligned} & E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q[\tilde{\tau}_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q \left[\int_0^{\tilde{\tau}_i} |(\tilde{X}_t - \hat{X}_t)(\tilde{X}_t + \hat{X}_t)| dt \right] \right] \\ & \leq C E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} Z_{T_i}^{-1} \frac{T_{i+1} - T_i}{E_{\mathcal{F}_{T_i}}^Q[\tilde{\tau}_i]} \varepsilon^{-2} E_{\mathcal{F}_{T_i}}^Q [\tilde{\tau}_i^3 + \tilde{\tau}_i^2 \varepsilon] \right] \\ & \leq C E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} Z_{T_i}^{-1} (T_{i+1} - T_i) \frac{\varepsilon^{-2} \overline{f}_\varepsilon^{0,3}(\underline{a}_{T_i}, \overline{a}_{T_i}) + \varepsilon^{-1} \overline{f}_\varepsilon^{0,2}(\underline{a}_{T_i}, \overline{a}_{T_i})}{\underline{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \overline{a}_{T_i})} \right], \quad (20) \end{aligned}$$

where C is a constant which does not depend on ε . Using Lemma 5 and the fact that $\alpha > 1$, we get

$$\sup_{\frac{1}{K} \leq a, b \leq K} \frac{\varepsilon^{-2} \overline{f}_\varepsilon^{0,3}(a, b) + \varepsilon^{-1} \overline{f}_\varepsilon^{0,2}(a, b)}{\underline{f}_\varepsilon^{0,1}(a, b)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

This, together with the fact that $E^Q[\sup_{0 \leq t \leq T} Z_t^{-1}] < \infty$, enables us to apply the dominated convergence theorem and conclude that (20) goes to zero.

Finally, we have that

$$\begin{aligned} S_1 & \leq \limsup_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} (T_{i+1} - T_i) \frac{\varepsilon^{-2-\alpha} \overline{f}_\varepsilon^{2,1}(\underline{a}_{T_i}, \overline{a}_{T_i})}{\varepsilon^{-\alpha} \underline{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \overline{a}_{T_i})} \right] \\ S_1 & \geq \limsup_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=0}^{\infty} 1_{T_i \leq T} A_{T_i} Z_{T_i}^{-1} (T_{i+1} - T_i) \frac{\varepsilon^{-2-\alpha} \underline{f}_\varepsilon^{2,1}(\underline{a}_{T_i}, \overline{a}_{T_i})}{\varepsilon^{-\alpha} \overline{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \overline{a}_{T_i})} \right]. \end{aligned}$$

Using a Riemann-sum type argument and the dominated convergence theorem together with the fact that

$$\frac{\varepsilon^{-2-\alpha} \overline{f}_\varepsilon^{2,1}(a, b)}{\varepsilon^{-\alpha} \underline{f}_\varepsilon^{0,1}(a, b)} \quad \text{and} \quad \frac{\varepsilon^{-2-\alpha} \underline{f}_\varepsilon^{2,1}(a, b)}{\varepsilon^{-\alpha} \overline{f}_\varepsilon^{0,1}(a, b)}$$

converge to

$$\frac{f^{*,2,1}(a,b)}{f^{*,0,1}(a,b)}$$

uniformly on $\frac{1}{K} \leq a, b \leq K$, we obtain that

$$S_1 = E^Q \left[\int_0^T A_t Z_t^{-1} \frac{f^{*,2,1}(\underline{a}_t, \bar{a}_t)}{f^{*,0,1}(\underline{a}_t, \bar{a}_t)} dt \right] = E \left[\int_0^T A_t \frac{f^{*,2,1}(\underline{a}_t, \bar{a}_t)}{f^{*,0,1}(\underline{a}_t, \bar{a}_t)} dt \right].$$

5 Preliminaries for the proof of Theorem 2

In this section, we prove some technical lemmas concerning the uniform integrability of the hitting time counts and the overshoots, which are needed for the proof of Theorem 2.

Lemma 6. *Under the assumptions (H_X) and (C_1) , for all $\beta \in [0, \alpha)$ and $\varepsilon > 0$,*

$$\begin{aligned} E_{\mathcal{F}_{T_i}}[|X_{T_{i+1}} - X_{T_i}|^\beta] &\leq c\varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} |b_s| ds \right] \\ &\quad + c\varepsilon^{\beta-\alpha} \max\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{\beta \vee (2-\alpha)} \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{(\beta-2) \wedge (-\alpha)} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \right], \end{aligned} \quad (21)$$

provided that the right-hand side has finite expectation.

Corollary 1. *Assume (H_X) and (C_1) . Then for all $\varepsilon > 0$,*

$$\varepsilon^\alpha \leq c\varepsilon^{\alpha-1} \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{-1} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} |b_s| ds \right] + c \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{-\alpha} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \right],$$

provided that the right-hand side has finite expectation.

Proof of Corollary. Apply Lemma 6 with $\beta' = 0$, $\underline{a}'_{T_i} = \bar{a}'_{T_i} = \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}$; then multiply both sides of (21) by ε^α and use the fact that the hitting time of the new barrier is smaller than T_{i+1} . \square

Proof of Lemma 6. First, observe that from the condition (C_1) we easily deduce by integration by parts that

$$\int_{x < |z| \leq 1} |z| K_t(z) \nu(dz) < Cx^{1-\alpha} \quad \text{and} \quad \int_{|z| \leq x} z^2 K_t(z) \nu(dz) < Cx^{2-\alpha} \quad (22)$$

for all $x > 0$, for a different constant $C < \infty$.

For this proof, let

$$f(x) := x^2 1_{0 \leq x \leq 2\bar{a}_{T_i}\varepsilon} (2\bar{a}_{T_i}\varepsilon)^{\beta-2} + |x|^\beta 1_{x > 2\bar{a}_{T_i}\varepsilon} + x^2 1_{-2\underline{a}_{T_i}\varepsilon \leq x \leq 0} (2\underline{a}_{T_i}\varepsilon)^{\beta-2} + |x|^\beta 1_{x < -2\underline{a}_{T_i}\varepsilon}.$$

By Itô formula,

$$\begin{aligned}
2^{\beta-2} E_{\mathcal{F}_{T_i}} [|X_{T_{i+1}} - X_{T_i}|^\beta] &\leq E_{\mathcal{F}_{T_i}} [f(X_{T_{i+1}} - X_{T_i})] = E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} f'(X_s - X_{T_i}) b_s ds \right] \\
&+ E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} \left\{ f(X_s + z - X_{T_i}) - f(X_s - X_{T_i}) - f'(X_s - X_{T_i}) z 1_{|z| \leq 1} \right\} \mu(ds \times dz) \right] \\
&+ E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{\mathbb{R}} \left\{ f(X_{s-} + z - X_{T_i}) - f(X_s - X_{T_i}) \right\} (M - \mu)(ds \times dz) \right].
\end{aligned} \tag{23}$$

The first term in the right-hand side satisfies

$$E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} f'(X_s - X_{T_i}) b_s ds \right] \leq (2\varepsilon)^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} |b_s| ds \right].$$

For the second term, we denote $A_s := \{z : X_s + z - X_{T_i} \in (-2\underline{a}_{T_i}\varepsilon, 2\bar{a}_{T_i}\varepsilon)\}$ and decompose it into two terms:

$$\begin{aligned}
&E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s^c} \left\{ f(X_s + z - X_{T_i}) - f(X_s - X_{T_i}) - f'(X_s - X_{T_i}) z 1_{|z| \leq 1} \right\} \mu(ds \times dz) \right] \\
&\leq C E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s^c} \left\{ |z|^\beta + \varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} |z| 1_{|z| \leq 1} \right\} \mu(ds \times dz) \right] \\
&\leq C E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \int_{(-\underline{a}_{T_i}\varepsilon, \bar{a}_{T_i}\varepsilon)^c} \left\{ |z|^\beta + \varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} |z| 1_{|z| \leq 1} \right\} K_s(dz) \right] \\
&\leq C \varepsilon^{\beta-\alpha} \left\{ \max\{\underline{a}_{T_i}^{\beta-\alpha}, \bar{a}_{T_i}^{\beta-\alpha}\} + \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} \max\{\underline{a}_{T_i}^{1-\alpha}, \bar{a}_{T_i}^{1-\alpha}\} \right\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \right]
\end{aligned}$$

and

$$\begin{aligned}
&E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s} \left\{ f(X_s + z - X_{T_i}) - f(X_s - X_{T_i}) - f'(X_s - X_{T_i}) z 1_{|z| \leq 1} \right\} \mu(ds \times dz) \right] \\
&= E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s} \left\{ f(X_s + z - X_{T_i}) - f(X_s - X_{T_i}) - f'(X_s - X_{T_i}) z \right\} \mu(ds \times dz) \right] \\
&- E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s} \left\{ f'(X_s - X_{T_i}) z 1_{|z| > 1} \right\} \mu(ds \times dz) \right] \\
&\leq E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s} \left\{ \frac{1}{2} \int_0^z f''(X_s - X_{T_i} + x)(z - x) dx \right\} \mu(ds \times dz) \right] \\
&- E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s} \left\{ f'(X_s - X_{T_i}) z 1_{|z| > 1} \right\} \mu(ds \times dz) \right],
\end{aligned}$$

which is smaller than

$$\begin{aligned}
& C\varepsilon^{\beta-2} \max\{\underline{a}_{T_i}^{\beta-2}, \bar{a}_{T_i}^{\beta-2}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{-3\underline{a}_{T_i}\varepsilon}^{3\bar{a}_{T_i}\varepsilon} z^2 \mu(ds \times dz) \right] \\
& + C\varepsilon^{\beta-1} \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{-3\underline{a}_{T_i}\varepsilon}^{3\bar{a}_{T_i}\varepsilon} z^2 \mu(ds \times dz) \right] \\
& \leq C\varepsilon^{\beta-\alpha} (\max\{\underline{a}_{T_i}^{\beta-2}, \bar{a}_{T_i}^{\beta-2}\} + \varepsilon \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\}) \\
& \quad \times \max\{\underline{a}_{T_i}^{2-\alpha}, \bar{a}_{T_i}^{2-\alpha}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \right].
\end{aligned}$$

Assembling the terms and doing some simple estimations yields the statement of the lemma, provided we can show that the third term in the right-hand side of (23) is equal to zero. Splitting it, once again, in two parts, we then get

$$\begin{aligned}
& E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{A_s^c} \left| f(X_{s-} + z - X_{T_i}) - f(X_s - X_{T_i}) \right| \mu(ds \times dz) \right] \\
& \leq C E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{(-\infty, -\underline{a}_{T_i}\varepsilon) \cup (\bar{a}_{T_i}\varepsilon, \infty)} |z|^\beta \mu(ds \times dz) \right] \\
& \leq C \max\{\underline{a}_{T_i}^{\beta-\alpha}, \bar{a}_{T_i}^{\beta-\alpha}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \right],
\end{aligned}$$

and for the other term,

$$\begin{aligned}
& E_{\mathcal{F}_{T_i}} \left[\left(\int_{T_i}^{T_{i+1}} \int_{A_s} \frac{f(X_{s-} + z - X_{T_i}) - f(X_s - X_{T_i})}{\max\{\underline{a}_{T_i}^{\beta-2}, \bar{a}_{T_i}^{\beta-2}\}} (M - \mu)(ds \times dz) \right)^2 \right] \\
& \leq \varepsilon^{2\beta-4} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \int_{-3\underline{a}_{T_i}\varepsilon}^{3\bar{a}_{T_i}\varepsilon} z^2 \mu(ds \times dz) \right] \\
& \leq \varepsilon^{2\beta-2-\alpha} \max\{\underline{a}_{T_i}^{2-\alpha}, \bar{a}_{T_i}^{2-\alpha}\} E_{\mathcal{F}_{T_i}} \left[\int_{T_i}^{T_{i+1}} \lambda_s ds \right].
\end{aligned}$$

Using the fact that both these terms have finite expectation by the assumption of the Lemma, we can now apply standard martingale arguments to show that the third term in (23) is equal to zero. \square

Lemma 7. Assume (H_X) , (C_1) and (C_2) . Let $\{\tau_n\}$ be a sequence of stopping times converging to T from below. Then there exists $\varepsilon^* > 0$ such that

$$\sup_{0 < \varepsilon < \varepsilon^*} E \left[\left(\varepsilon^{\alpha-\beta} \sum_{i=1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta \right)^{1+\delta} \right] < \infty. \quad (24)$$

and

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \downarrow 0} E \left[\left(\varepsilon^{\alpha-\beta} \sum_{i=N_{\tau_n}^\varepsilon+1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta \right)^{1+\delta} \right] = 0. \quad (25)$$

Proof. In this proof, we shall use the notation

$$\begin{aligned}\bar{\Lambda}_t = & \sup_{0 \leq s \leq T} (\max\{\underline{a}_s^{\beta-1}, \bar{a}_s^{\beta-1}\}^{1+\delta} + \max\{\underline{a}_s^{(1+\delta)\beta-1}, \bar{a}_s^{(1+\delta)\beta-1}\})|b_t|^{1+\delta} + \\ & \sup_{0 \leq s \leq T} \max\{\underline{a}_s, \bar{a}_s\}^{(\beta \vee (2-\alpha))(1+\delta)} \min\{\underline{a}_s, \bar{a}_s\}^{((\beta-2) \wedge (-\alpha))(1+\delta)} \lambda_t^{1+\delta}.\end{aligned}$$

We decompose the process to be estimated as

$$\begin{aligned}\sum_{i=1}^n |X_{T_i} - X_{T_{i-1}}|^\beta &= M_n^1 + M_n^2 + Z_n, \\ M_n^1 &= \sum_{i=1}^n \left\{ |X_{T_i} - X_{T_{i-1}}|^\beta - E_{\mathcal{F}_{T_{i-1}}} [|X_{T_i} - X_{T_{i-1}}|^\beta] \right\}, \\ M_n^2 &= \sum_{i=1}^n E_{\mathcal{F}_{T_{i-1}}} [|X_{T_i} - X_{T_{i-1}}|^\beta] \left\{ 1 - \frac{\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds}{E_{\mathcal{F}_{T_{i-1}}} \left[\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds \right]} \right\}, \\ Z_n &= \sum_{i=1}^n E_{\mathcal{F}_{T_{i-1}}} [|X_{T_i} - X_{T_{i-1}}|^\beta] \frac{\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds}{E_{\mathcal{F}_{T_{i-1}}} \left[\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds \right]},\end{aligned}$$

where we write

$$\Lambda_s^{T_i} := \varepsilon^{\alpha-1} \max\{\underline{a}_{T_i}^{\beta-1}, \bar{a}_{T_i}^{\beta-1}\} |b_s| + \max\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{\beta \vee (2-\alpha)} \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{(\beta-2) \wedge (-\alpha)} \lambda_s.$$

The processes M^1 and M^2 are martingales with respect to the discrete filtration $\mathcal{F}_n^d := \mathcal{F}_{T_n}$. Note that for every \mathcal{F} -stopping time $\tau \leq T$, N_τ^ε is an \mathcal{F}^d -stopping time. The Burkholder inequality for a discrete-time martingale M then writes

$$\begin{aligned}E[|M_{N_T^\varepsilon} - M_{N_\tau^\varepsilon}|^{1+\delta}] &\leq CE \left[\left(\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} (M_i - M_{i-1})^2 \right)^{\frac{1+\delta}{2}} \right] \\ &\leq CE \left[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} |M_i - M_{i-1}|^{1+\delta} \right],\end{aligned}$$

and therefore,

$$\begin{aligned}&E[|\varepsilon^{\alpha-\beta} (M_{N_T^\varepsilon}^1 - M_{N_\tau^\varepsilon}^1)|^{1+\delta}] \\ &\leq C\varepsilon^{(\alpha-\beta)(1+\delta)} E \left[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} \left| |X_{T_i} - X_{T_{i-1}}|^\beta - E_{\mathcal{F}_{T_{i-1}}} [|X_{T_i} - X_{T_{i-1}}|^\beta] \right|^{1+\delta} \right] \\ &\leq C\varepsilon^{(\alpha-\beta)(1+\delta)} E \left[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} E_{\mathcal{F}_{T_{i-1}}} [|X_{T_i} - X_{T_{i-1}}|^{\beta(1+\delta)}] \right].\end{aligned}$$

By Lemma 6, this is smaller than

$$\begin{aligned}
& CE \left[\varepsilon^{\alpha(1+\delta)-1} \sup_{0 \leq s \leq T} \min\{\underline{a}_s, \bar{a}_s\}^{\beta'-1} \int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} |b_s| ds \right. \\
& \quad \left. + \varepsilon^{\alpha\delta} \sup_{0 \leq s \leq T} \max\{\underline{a}_s, \bar{a}_s\}^{\beta' \vee (2-\alpha)} \min\{\underline{a}_s, \bar{a}_s\}^{(\beta'-2) \wedge (-\alpha)} \int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \lambda_s ds \right] \\
& \leq C\varepsilon^{\alpha\delta} \left(E \left[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \bar{\Lambda}_s ds \right] + E \left[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \bar{\Lambda}_s ds \right]^{\frac{1}{1+q}} \right),
\end{aligned}$$

with $\beta' = \beta(1 + \delta)$, where the last estimate can be obtained, e.g. by Hölder inequality.

Similarly, the process M^2 satisfies

$$\begin{aligned}
E[|\varepsilon^{\alpha-\beta}(M_{N_T^\varepsilon}^2 - M_{N_\tau^\varepsilon}^2)|^{1+\delta}] & \leq CE \left[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} \left\{ \int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds - E_{\mathcal{F}_{T_{i-1}}} \left[\int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds \right] \right\}^{1+\delta} \right] \\
& \leq CE \left[\sum_{i=N_\tau^\varepsilon+1}^{N_T^\varepsilon} \left\{ \int_{T_{i-1}}^{T_i} \Lambda_s^{T_{i-1}} ds \right\}^{1+\delta} \right] \leq CE \left[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} (\Lambda_s^{\eta_s})^{1+\delta} ds \right] \leq CE \left[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \bar{\Lambda}_s ds \right].
\end{aligned}$$

The process Z can be treated along the same lines as well, since by Lemma 6,

$$E[|\varepsilon^{\alpha-\beta}(Z_{N_T^\varepsilon} - Z_{N_\tau^\varepsilon})|^{1+\delta}] \leq CE \left[\left\{ \int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \Lambda_s^{\eta_s} ds \right\}^{1+\delta} \right] \leq CE \left[\int_{T_{N_\tau^\varepsilon}}^{T_{N_T^\varepsilon}} \bar{\Lambda}_s ds \right].$$

The three expressions above are uniformly bounded by the assumption of the lemma, proving (24). To show (25), observe that

$$E \left[\int_{T_{N_{\tau_n}^\varepsilon}}^{T_{N_T^\varepsilon}} \bar{\Lambda}_s ds \right] \leq E \left[\int_{\tau_n}^T \bar{\Lambda}_s ds \right] + E \left[\sup_{i: T_i \leq T} \int_{T_{i-1}}^{T_i} \bar{\Lambda}_s ds \right].$$

The first term does not depend on ε and converges to zero as $n \rightarrow \infty$ by the assumption of the lemma and the dominated convergence. For the second term, we use Lemma 3 and the absolute continuity of the integral. \square

In the case $\beta = 0$, the assumption (C_2) can be somewhat simplified.

Lemma 8. *Assume (H_X) , (C_1) and (C'_2) . Let $\{\tau_n\}$ be a sequence of stopping times converging to T from below. Then there exists $\varepsilon^* > 0$ such that*

$$\sup_{0 < \varepsilon < \varepsilon^*} E[(\varepsilon^\alpha N_T^\varepsilon)^{1+\delta}] < \infty.$$

and

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \downarrow 0} E \left[\left(\varepsilon^\alpha (N_T^\varepsilon - N_{\tau_n}^\varepsilon) \right)^{1+\delta} \right] = 0.$$

Proof. We follow the proof of Lemma 7, taking $\beta = 0$,

$$\Lambda_s^{T_i} := \varepsilon^{\alpha-1} \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{-1} |b_s| + \min\{\underline{a}_{T_i}, \bar{a}_{T_i}\}^{-\alpha} \lambda_s,$$

and using Corollary 1 instead of Lemma 6. \square

6 Proof of Theorem 2

Step 1. Reduction to the case of bounded coefficients. As before, we start with the localization procedure.

Lemma 9. *Assume that (12) holds under the assumptions (H_Y) , (H_X) and (H'_1) and (13) holds under the assumptions (H_Y) , (H_X) and (H'_ρ) . Then Theorem 2 holds.*

Proof. The arguments related to the localization of Z are the same or very similar to those in Lemma 2 and so they are omitted. With the same notation as in the proof of this Lemma, and using (25) in the first equality we then get, for $0 \leq \beta < \alpha$:

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E \left[\sum_{i=1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta \right] &= \lim_{n \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E \left[\sum_{i=1}^{N_{\gamma_n}^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta \right] \\ &= \lim_{n \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E \left[\sum_{i \geq 1: T_i^n \leq \gamma_n} |X_{T_i}^n - X_{T_{i-1}}^n|^\beta \right] \\ &= \lim_{n \rightarrow \infty} E \left[\int_0^{\gamma_n} \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right] = E \left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{g(\underline{a}_t, \bar{a}_t)} dt \right], \end{aligned}$$

where the assumption of the lemma was used to pass from the second to the third line.

Step 2. Change of probability measure. The goal of this step is to show that

$$\begin{aligned} S_2 &:= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E \left[\sum_{i=1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta \right] \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} |X_{T_i} - X_{T_{i-1}}|^\beta \right]. \quad (26) \end{aligned}$$

For the right-hand side to be well defined we extend the processes λ , b , \underline{a} , \bar{a} by arbitrary constant values beyond T and define the process X for $t \geq T$ accordingly. The case $\beta = 0$ being straightforward, we assume that $\beta > 0$.

To prove (26), it is enough to show that

$$\lim_{\varepsilon \downarrow 0} E^Q \left[\varepsilon^{\alpha-\beta} \sum_{i=1}^{\infty} 1_{T_i \leq T} (Z_{T_i}^{-1} - Z_{T_{i-1}}^{-1}) |X_{T_i} - X_{T_{i-1}}|^\beta \right] = 0 \quad (27)$$

$$\text{and } \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[Z_{T_{N_T^\varepsilon}}^{-1} |X_{T_{N_T^\varepsilon+1}} - X_{T_{N_T^\varepsilon}}|^\beta \right] = 0. \quad (28)$$

The second term can be shown to converge to zero using Lemma 6. For the first term, for $1 < \kappa < \frac{\alpha\rho}{\alpha+\beta\rho}$, Hölder inequality yields

$$\begin{aligned} & E^Q \left[\left(\varepsilon^{\alpha-\beta} \sum_{i=1}^{\infty} 1_{T_i \leq T} (Z_{T_i}^{-1} - Z_{T_{i-1}}^{-1}) |X_{T_i} - X_{T_{i-1}}|^\beta \right)^\kappa \right] \\ & \leq E^Q \left[\sup_{0 \leq t \leq T} Z_t^{-\rho} \right]^{\frac{\kappa}{\rho}} E^Q \left[\left(\varepsilon^{\alpha-\beta} \sum_{i=1}^{\infty} 1_{T_i \leq T} |X_{T_i} - X_{T_{i-1}}|^\beta \right)^{\frac{\kappa\rho}{\rho-\kappa}} \right]^{\frac{\rho-\kappa}{\rho}}, \end{aligned}$$

which is bounded by a constant for ε sufficiently small by Lemma 7 (applied under Q) (the assumptions are satisfied because we are working under H'_ρ and therefore all coefficients are bounded). Therefore, the expression under the expectation in (27) is uniformly integrable under Q as $\varepsilon \downarrow 0$. On the other hand, by Cauchy-Schwarz inequality,

$$\begin{aligned} & \varepsilon^{\alpha-\beta} \sum_{i=1}^{\infty} 1_{T_i \leq T} |Z_{T_i}^{-1} - Z_{T_{i-1}}^{-1}| |X_{T_i} - X_{T_{i-1}}|^\beta \\ & \leq C \varepsilon^{\frac{\alpha-\beta}{2}} \left(\sum_{i=1}^{N_T^\varepsilon} (Z_{T_i}^{-1} - Z_{T_{i-1}}^{-1})^2 \right)^{\frac{1}{2}} \sup_{0 \leq t \leq T} |X_t|^\beta \left(\varepsilon^{\alpha-\beta} \sum_{i=1}^{N_T^\varepsilon} |X_{T_i} - X_{T_{i-1}}|^\beta \right)^{\frac{1}{2}}. \end{aligned}$$

Since Z^{-1} has finite quadratic variation, and the last factor is uniformly integrable under Q by Lemma 7, due to the first deterministic factor, the whole expression converges to zero in probability and (27) follows.

Step 3. Using the same notation as in the proof of Theorem 1 (Step 3), we have,

$$\begin{aligned} & \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |\tilde{X}_{\tilde{\tau}_i}|^\beta}{E_{\mathcal{F}_{T_{i-1}}}^Q [T_i - T_{i-1}]} \right] \\ & = \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |\tilde{X}_{\tilde{\tau}_i}|^\beta}{E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i]} \right] + R^\varepsilon, \end{aligned}$$

where one can show, using first Lemma 6 and then exactly the same arguments as in the proof of Theorem 1, that $R^\varepsilon \rightarrow 0$ as $\varepsilon \downarrow 0$. Then, from the previous

step,

$$\begin{aligned}
S_2 &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |X_{T_i} - X_{T_{i-1}}|^\beta}{E_{\mathcal{F}_{T_{i-1}}}^Q [T_i - T_{i-1}]} \right] \\
&= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |\tilde{X}_{\tilde{\tau}_i}|^\beta}{E_{\mathcal{F}_{T_{i-1}}}^Q [T_i - T_{i-1}]} \right] \\
&= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |\tilde{X}_{\tilde{\tau}_i}|^\beta}{E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i]} \right].
\end{aligned}$$

Our next goal is to replace $\tilde{X}_{\tilde{\tau}_i}$ with $\hat{X}_{\hat{\tau}_i}$ in the above expression, where $\hat{\tau}_i = \inf\{t \geq 0 : \hat{X}_t \notin [-\underline{a}_{T_i}\varepsilon, \bar{a}_{T_i}\varepsilon]\}$. Let $a = \min(\underline{a}_{T_i}, \bar{a}_{T_i})$ and define

$$f(x) = (\varepsilon a)^\beta \frac{(\beta - \varepsilon a) \left(\frac{x}{\varepsilon a}\right)^2 + 2 - \beta}{2 - \varepsilon a} 1_{|x| < \varepsilon a} + |x|^\beta 1_{|x| > \varepsilon a}.$$

f is a twice differentiable function satisfying

$$|f'(x)| \leq C\varepsilon^{\beta-1} \quad \text{and} \quad |f''(x)| \leq C\varepsilon^{\beta-2} \quad (29)$$

and hence the Itô formula can be applied. Then,

$$\left| E_{\mathcal{F}_{T_{i-1}}}^Q [|\tilde{X}_{\tilde{\tau}_i}|^\beta - |\hat{X}_{\hat{\tau}_i}|^\beta] \right| \leq \left| E_{\mathcal{F}_{T_{i-1}}}^Q [f(\tilde{X}_{\tilde{\tau}_i}) - f(\hat{X}_{\hat{\tau}_i})] \right| + \left| E_{\mathcal{F}_{T_{i-1}}}^Q [f(\hat{X}_{\hat{\tau}_i}) - f(\hat{X}_{\hat{\tau}_i})] \right|.$$

By definition of \tilde{X} and \hat{X} and because all coefficients are bounded, the first term satisfies

$$\left| E_{\mathcal{F}_{T_{i-1}}}^Q [f(\tilde{X}_{\tilde{\tau}_i}) - f(\hat{X}_{\hat{\tau}_i})] \right| \leq C\varepsilon^{\beta-1} E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i].$$

For the second term, we use the Itô formula:

$$\begin{aligned}
&E_{\mathcal{F}_{T_{i-1}}}^Q [f(\hat{X}_{\hat{\tau}_i}) - f(\hat{X}_{\hat{\tau}_i})] \\
&= E_{\mathcal{F}_{T_{i-1}}}^Q \left[\int_{\hat{\tau}_i \wedge \tilde{\tau}_i}^{\hat{\tau}_i \vee \tilde{\tau}_i} \int_{\mathbb{R}} \{f(\hat{X}_s + z) - f(\hat{X}_s) - z 1_{|z| \leq 1} f'(\hat{X}_s)\} \nu(dz) ds \right] \\
&+ E_{\mathcal{F}_{T_{i-1}}}^Q \left[\int_{\hat{\tau}_i \wedge \tilde{\tau}_i}^{\hat{\tau}_i \vee \tilde{\tau}_i} \int_{\mathbb{R}} \{f(\hat{X}_{s-} + z) - f(\hat{X}_{s-})\} (\widehat{M}(ds \times dz) - \nu(dz) ds) \right],
\end{aligned}$$

where \widehat{M} is the jump measure of \hat{X} . It follows by standard arguments that the local martingale term has zero expectation. To deal with the first term we use the bounds (29) and decompose the integrand as follows:

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \{f(\hat{X}_s + z) - f(\hat{X}_s) - z 1_{|z| \leq 1} f'(\hat{X}_s)\} \nu(dz) \right| \\
&\leq C\varepsilon^{\beta-2} \int_{|z| \leq \varepsilon} z^2 \nu(dz) + C\varepsilon^{\beta-1} \int_{|z| > \varepsilon} |z| \nu(dz) \leq C\varepsilon^{\beta-\alpha},
\end{aligned}$$

so that finally

$$\left| E_{\mathcal{F}_{T_{i-1}}}^Q [|\tilde{X}_{\tilde{\tau}_i}|^\beta - |\hat{X}_{\hat{\tau}_i}|^\beta] \right| \leq C\varepsilon^{\beta-1} E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i] + C\varepsilon^{\beta-\alpha} E_{\mathcal{F}_{T_{i-1}}}^Q [|\tilde{\tau}_i - \hat{\tau}_i|].$$

Substituting this estimate into the formula for S_2 , we then get:

$$S_2 = \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |\hat{X}_{\hat{\tau}_i}|^\beta}{E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i]} \right] + \lim_{\varepsilon \downarrow 0} R^\varepsilon$$

with

$$\begin{aligned} |R^\varepsilon| &\leq C\varepsilon^{\alpha-1} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \right] \\ &\quad + CE^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{E_{\mathcal{F}_{T_{i-1}}}^Q |\tilde{\tau}_i - \hat{\tau}_i|}{E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i]} \right]. \end{aligned}$$

The first expectation is bounded (because λ is bounded and Z^{-1} is integrable, and therefore the first term converges to zero. For the second term, we observe (using the notation of the proof of Theorem 1, Step 4) that

$$E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i] \leq \bar{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \bar{a}_{T_i})$$

$$\text{and } E_{\mathcal{F}_{T_{i-1}}}^Q |\tilde{\tau}_i - \hat{\tau}_i| \leq E_{\mathcal{F}_{T_{i-1}}}^Q (\hat{\tau}_2 \wedge \hat{\tau}^j - \hat{\tau}_1) \leq \bar{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \bar{a}_{T_i}) - \underline{f}_\varepsilon^{0,1}(\underline{a}_{T_i}, \bar{a}_{T_i}).$$

In view of Lemma 5 we then conclude that the second term converges to zero as well. Finally, we have shown that

$$S_2 = \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-\beta} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{u_\varepsilon^\beta(\underline{a}_{T_{i-1}}, \bar{a}_{T_{i-1}})}{E_{\mathcal{F}_{T_{i-1}}}^Q [\tilde{\tau}_i]} \right],$$

where u_ε^β is a deterministic function defined by

$$u_\varepsilon^\beta(a, b) = E[|\hat{X}_{\hat{\tau}}|^\beta], \quad \hat{\tau} = \inf\{t \geq 0 : \hat{X}_t \notin (-a\varepsilon, b\varepsilon)\}.$$

Similarly to the last step of the proof of Theorem 1, we can now write

$$\begin{aligned} S_2 &\leq \limsup_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{\varepsilon^{-\beta} u_\varepsilon^\beta(\underline{a}_{T_{i-1}}, \bar{a}_{T_{i-1}})}{\varepsilon^{-\alpha} \bar{f}_\varepsilon^{0,1}(\underline{a}_{T_{i-1}}, \bar{a}_{T_{i-1}})} \right], \\ S_2 &\geq \limsup_{\varepsilon \downarrow 0} E^Q \left[\sum_{i=1}^{\infty} 1_{T_{i-1} \leq T} Z_{T_{i-1}}^{-1} \lambda_{T_{i-1}} (T_i - T_{i-1}) \frac{\varepsilon^{-\beta} u_\varepsilon^\beta(\underline{a}_{T_{i-1}}, \bar{a}_{T_{i-1}})}{\varepsilon^{-\alpha} \bar{f}_\varepsilon^{0,1}(\underline{a}_{T_{i-1}}, \bar{a}_{T_{i-1}})} \right]. \end{aligned}$$

Using Lemma (11) we obtain uniform convergence of

$$\frac{\varepsilon^{-\beta} u_\varepsilon^\beta(a, b)}{\varepsilon^{-\alpha} \bar{f}_\varepsilon^{0,1}(a, b)}$$

towards $\frac{u^\beta(a,b)}{f^{*,0,1}(a,b)}$ and conclude that

$$S_2 = E^Q \left[\int_0^T \lambda_t Z_t^{-1} \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{f^{*,0,1}(\underline{a}_t, \bar{a}_t)} dt \right] = E \left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \bar{a}_t)}{f^{*,0,1}(\underline{a}_t, \bar{a}_t)} dt \right].$$

□

Appendix

Some computations for stable processes

Proposition 1. *Let X be a symmetric α -stable process on \mathbb{R} with characteristic function $E[e^{iuX_t}] = e^{-t\sigma|u|^\alpha}$, and $\tau_{a,b} = \inf\{t \geq 0 : X_t \notin (-a, b)\}$. Then,*

$$g(a, b) := E \left[\int_0^{\tau_{a,b}} X_t^2 dt \right] = \frac{\alpha(ab)^{1+\frac{\alpha}{2}}}{2\sigma\Gamma(3+\alpha)} \left\{ \left(\frac{a}{b} + \frac{b}{a} \right) \left(1 + \frac{\alpha}{2} \right) - \alpha \right\}.$$

The proof of this result is based on the following lemma, where we consider the exit time from the interval $[-1, 1]$ by a process starting from x .

Lemma 10. *Let X be as above and $\tau_1 = \inf\{t \geq 0 : X_t \notin (-1, 1)\}$. Then*

$$g(x) := E^x \left[\int_0^{\tau_1} X_t^2 dt \right] = \frac{1}{\sigma} \frac{2(1-x^2)^{\frac{\alpha}{2}} \{x^2 + \frac{\alpha}{2}\}}{\Gamma(3+\alpha)} 1_{x \in (-1, 1)}.$$

Proof of lemma. Without loss of generality, we let $\sigma = 1$ in this proof. Let $\hat{g}(u) = \int_{\mathbb{R}} e^{iux} g(x) dx$. Using the arguments similar to the ones in [12], one can show that the function g satisfies the equation $\mathcal{L}^\alpha g_A(x) = -x^2$ on $x \in (-1, 1)$ with the boundary condition $g(x) = 0$ on $x \notin (-1, 1)$, where L^α is the fractional Laplace operator

$$L^\alpha f(x) = \int_{\mathbb{R}} (f(x+y) - f(x) - yf'(x)) \frac{dy}{|y|^{1+\alpha}}.$$

Moreover, the function \hat{g} satisfies the system of integral equations

$$\begin{aligned} \frac{1}{\pi} \int_0^\infty \hat{g}(u) |u|^\alpha \cos(ux) du &= x^2, \quad |x| < 1, \\ \frac{1}{\pi} \int_0^\infty \hat{g}(u) \cos(ux) du &= 0, \quad |x| > 1. \end{aligned}$$

Let $\hat{g}_1(u) = u^{-\frac{1+\alpha}{2}} J_{\frac{1+\alpha}{2}}(u)$ and $\hat{g}_2(u) = u^{-\frac{3+\alpha}{2}} J_{\frac{3+\alpha}{2}}(u)$. Then, from [13, Integral

6.699.2], we get:

$$\int_0^\infty \hat{g}_1(u) \cos(ux) du = \int_0^\infty \hat{g}_2(u) \cos(ux) du = 0, \quad |x| > 1, \quad (30)$$

$$\int_0^\infty \hat{g}_1(u) |u|^\alpha \cos(ux) du = 2^{\frac{\alpha-1}{2}} \Gamma\left(\frac{1+\alpha}{2}\right) \quad |x| < 1, \quad (31)$$

$$\int_0^\infty \hat{g}_2(u) |u|^\alpha \cos(ux) du = 2^{\frac{\alpha-3}{2}} \Gamma\left(\frac{1+\alpha}{2}\right) (1 - (1+\alpha)x^2), \quad |x| < 1, \quad (32)$$

$$\int_0^\infty \hat{g}_1(u) \cos(ux) du = 2^{-\frac{\alpha+1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)} (1-x^2)^{\frac{\alpha}{2}} \quad |x| < 1, \quad (33)$$

$$\int_0^\infty \hat{g}_2(u) \cos(ux) du = 2^{-\frac{\alpha+3}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+4}{2}\right)} (1-x^2)^{1+\frac{\alpha}{2}}, \quad |x| < 1. \quad (34)$$

From (30–32),

$$\hat{g}(u) = \pi \frac{\hat{g}_1(u) - 2\hat{g}_2(u)}{2^{\frac{\alpha-1}{2}} \Gamma\left(\frac{1+\alpha}{2}\right) (1+\alpha)}.$$

To conclude, we compute the inverse Fourier transform of \hat{g} from (33–34). \square

Proof of the proposition. Once again, we set $\sigma = 1$ without loss of generality. Recall a result of Blumenthal, Gettoor and Ray [4]: the law of a symmetric stable process starting from the point x with $|x| < 1$ and observed at time τ_1 has density given by

$$\mu(x, y) = \frac{1}{\pi} \sin \frac{\pi\alpha}{2} (1-x^2)^{\frac{\alpha}{2}} (y^2-1)^{-\frac{\alpha}{2}} |y-x|^{-1}, \quad |y| \geq 1.$$

By scaling property, we then deduce that the density of a symmetric stable process starting from zero, and observed at time $\tau_{a,b}$ is given by

$$\mu_{a,b}(z) = \frac{1}{\pi} \sin \frac{\pi\alpha}{2} (ab)^{\frac{\alpha}{2}} ((z+b)(z-a))^{-\frac{\alpha}{2}} \frac{1}{|z|}. \quad (35)$$

Similarly, from the preceding lemma, we easily deduce by scaling property that

$$g_A(x) := E^x \left[\int_0^{\tau_{A,A}} X_t^2 dt \right] = \frac{2(A^2 - x^2)^{\frac{\alpha}{2}} \left\{ x^2 + \frac{\alpha}{2} A^2 \right\}}{\Gamma(3+\alpha)} 1_{x \in (-A, A)}.$$

This function satisfies the equation $\mathcal{L}^\alpha g_A(x) = -x^2$ on $[-A, A]$ with the terminal condition $g_A(x) = 0$ on $x \notin [-A, A]$. Taking $A \geq (a, b)$, we then get by Itô formula

$$E[g_A(X_{\tau_{a,b}})] = g_A(0) - E \left[\int_0^{\tau_{a,b}} X_t^2 dt \right].$$

By symmetry, it is sufficient to prove the theorem for $a \geq b$. Taking $A = a$ in the above formula, we finally get

$$\begin{aligned} E \left[\int_0^{\tau_{a,b}} X_t^2 dt \right] &= \frac{\alpha a^{\alpha+2}}{\Gamma(3+\alpha)} - \int_b^a g_A(x) \mu_{a,b}(x) dx \\ &= \frac{\alpha a^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{2 \sin \frac{\pi\alpha}{2}}{\pi \Gamma(3+\alpha)} (ab)^{\frac{\alpha}{2}} \int_b^a (z^2 + \frac{\alpha}{2} a^2) \left(\frac{a-z}{z-b} \right)^{\frac{\alpha}{2}} \frac{dz}{z}. \end{aligned}$$

Computing the integral then yields the result. \square

Remark 3. Let us list here several other useful results which are already known from the literature or can be obtained with a simple computation. By a result of Gettoor [12]: under the assumptions of Proposition 1,

$$E^x[\tau_1] = \frac{1}{\sigma} \frac{2^{-\alpha} \Gamma(\frac{1}{2})}{\Gamma(\frac{2+\alpha}{2}) \Gamma(\frac{1+\alpha}{2})} (1-x^2)^{\frac{\alpha}{2}} = \frac{1}{\sigma} \frac{(1-x^2)^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)}.$$

By scaling property we then deduce that for general barriers

$$E[\tau_{a,b}] = \left(\frac{a+b}{2}\right)^{\alpha} E^{\frac{a-b}{a+b}}[\tau_1] = \frac{(ab)^{\frac{\alpha}{2}}}{\sigma \Gamma(1+\alpha)}. \quad (36)$$

Similarly, from (35), we easily get, for $\beta < \alpha$,

$$E[|X_{\tau_{a,b}}|^{\beta}] = \frac{\sin \frac{\pi\alpha}{2}}{\pi} (ab)^{\frac{\alpha}{2}} \int_0^{\infty} z^{-\alpha/2} (z+a+b)^{-\alpha/2} (|z+a|^{\beta-1} + |z+b|^{\beta-1}) dz, \quad (37)$$

and in the case $a = b$, this integral can be expressed in terms of special functions and is equal to

$$a^{\beta} \frac{2^{1-\frac{\alpha}{2}} \sin \frac{\pi\alpha}{2}}{\pi} B\left(1 - \frac{\alpha}{2}, \alpha - \beta\right) F\left(\frac{\alpha}{2}, 1 - \frac{\alpha}{2}, \frac{\alpha}{2} + 1 - \beta, \frac{1}{2}\right),$$

where B is the beta function and F is the hypergeometric function.

Convergence of rescaled Lévy processes

Lemma 11. *Let X be a Lévy process with characteristic triplet $(0, \nu, \gamma)$ with respect to the truncation function $h(x) = -1 \vee x \wedge 1$ with*

$$x^{\alpha} \nu((x, \infty)) \rightarrow c_+ \quad \text{and} \quad x^{\alpha} \nu((-\infty, -x)) \rightarrow c_- \quad \text{when} \quad x \rightarrow 0$$

for some $\alpha \in (1, 2)$ and constants $c_+ \geq 0$ and $c_- \geq 0$ with $c_+ + c_- > 0$. For $\varepsilon > 0$, define the process X^{ε} via $X_t^{\varepsilon} = \varepsilon^{-1} X_{\varepsilon^{\alpha} t}$. Then X^{ε} converges in law to a strictly α -stable Lévy process X^ with Lévy density*

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}. \quad (38)$$

Assume in addition that there exists $C < \infty$, such that for all $x > 0$,

$$\nu((-x, x)^c) < Cx^{-\alpha}$$

and for $a, b \in (0, \infty)$ and $\beta \in (0, \alpha)$, let

$$u_{\varepsilon}^{\beta}(a, b) = E[|X_{\tau_{\varepsilon}}^{\varepsilon}|^{\beta}], \quad \tau_{\varepsilon} = \inf\{t \geq 0 : X_t^{\varepsilon} \notin (-a, b)\}.$$

Then,

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon^\beta(a, b) = u^\beta(a, b)$$

uniformly on $(a, b) \in [K^{-1}, K]^2$ for all $K < \infty$, with

$$u^\beta(a, b) = E[|X_{\tau^*}^*|^\beta]$$

and $\tau^* = \inf\{t \geq 0 : X_t^* \notin (-a, b)\}$.

Proof. Part (i). From the Lévy-Khintchine formula it is easy to see that the characteristic triplet $(A^\varepsilon, \nu^\varepsilon, \gamma^\varepsilon)$ of X^ε is given by

$$\begin{aligned} A^\varepsilon &= 0 \\ \nu^\varepsilon(B) &= \varepsilon^\alpha \nu(\{x : x/\varepsilon \in B\}), \quad B \in \mathcal{B}(\mathbb{R}); \\ \gamma^\varepsilon &= \varepsilon^{\alpha-1} \left\{ \gamma + \int_{\mathbb{R}} \nu(dx) (\varepsilon h(x/\varepsilon) - h(x)) \right\}. \end{aligned}$$

Under the conditions of the Lemma, by Theorem VII.2.9 and Remark VII.2.10 in [15], in order to prove the convergence in law, we need to check (a) that

$$\gamma^\varepsilon \rightarrow -\frac{c_+ - c_-}{\alpha(\alpha - 1)},$$

where the right hand side is the third component of the characteristic triplet of the strictly stable process with Lévy density (38) with respect to the truncation function h , and (b) that $|x|^2 \wedge 1 \cdot \nu^\varepsilon(dx)$ converges weakly to $|x|^2 \wedge 1 \cdot \nu^*(dx)$. Since $\alpha > 1$ and h is bounded, for η sufficiently small, using integration by parts and the assumption of the Lemma, we obtain

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \gamma^\varepsilon &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-1} \int_{|x| \leq \eta} \nu(dx) (\varepsilon h(x/\varepsilon) - h(x)) \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-1} \left\{ \int_{-\eta}^{-\varepsilon} (-\varepsilon - x) \nu(dx) + \int_{\varepsilon}^{\eta} (\varepsilon - x) \nu(dx) \right\} \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-1} \left\{ \int_{-\eta}^{-\varepsilon} \nu([- \eta, x]) dx - \int_{\varepsilon}^{\eta} \nu([x, \eta]) dx \right\} \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-1} \left\{ \int_{-\eta}^{-\varepsilon} \nu((-\infty, x]) dx - \int_{\varepsilon}^{\eta} \nu([x, \infty)) dx \right\} \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha-1} \left\{ \int_{-\eta}^{-\varepsilon} \frac{c_-}{|x|^\alpha} dx - \int_{\varepsilon}^{\eta} \frac{c_+}{|x|^\alpha} dx \right\} = -\frac{c_+ - c_-}{\alpha(\alpha - 1)}. \end{aligned}$$

For the property (b), it is sufficient to show that for all $x \geq 0$,

$$\begin{aligned} \int_x^\infty |z|^2 \wedge 1 \cdot \nu^\varepsilon(dz) &\rightarrow \int_x^\infty |z|^2 \wedge 1 \cdot \nu^*(dz) \\ \text{and } \int_{-\infty}^{-x} |z|^2 \wedge 1 \cdot \nu^\varepsilon(dz) &\rightarrow \int_{-\infty}^{-x} |z|^2 \wedge 1 \cdot \nu^*(dz). \end{aligned}$$

This is done using integration by parts and the assumption of the Lemma as in the previous step.

Part (ii). First, similarly to the proof of Proposition 3 in [17], it is easy to show that $X_{\tau^\varepsilon}^\varepsilon$ converges in law to $X_{\tau^*}^*$ as $\varepsilon \downarrow 0$. To complete the proof of the lemma, it remains to show that for all $\beta \in (0, \alpha)$,

$$E[|X_{\tau^\varepsilon}^\varepsilon|^\beta]$$

is bounded uniformly on ε . From Lemma 6,

$$E[|X_{\tau^\varepsilon}^\varepsilon|^\beta] \leq C\varepsilon^{-\alpha}E[\tau^\varepsilon]$$

for some constant C which does not depend on ε . On the other hand, for ε small enough,

$$E[\tau^\varepsilon] \leq E[\inf\{t : |\Delta X_t| \geq \varepsilon(a+b)\}] = \frac{1}{\nu((-\varepsilon a, \varepsilon b)^c)} \leq C'\varepsilon^\alpha$$

for a different constant C' .

It remains to show that the convergence is uniform in a and b . First, using trajectorial continuity and uniform integrability as above, we show that $u^\beta(a, b)$ is continuous in (a, b) for $(a, b) \in [K^{-1}, K]^2$ and therefore also uniformly continuous on this set. Letting $\delta > 0$, we use this to choose ρ such that for all (a, b) and (a', b') belonging to $[K^{-1}, K]$, $|a - a'| + |b - b'| \leq \rho$ implies $|u^\beta(a, b) - u^\beta(a', b')| \leq \delta/2$.

Next, for every $\lambda > 0$,

$$u_\varepsilon^\beta(\lambda a, \lambda b) = \lambda^\beta u_{\varepsilon\lambda}^\beta(a, b),$$

which means that $u_\varepsilon^\beta(\lambda a, \lambda b)$ converges to $u^\beta(\lambda a, \lambda b)$ uniformly on $\lambda \in [\lambda_1, \lambda_2]$ for $0 < \lambda_1 < \lambda_2 < \infty$. For $K^{-1} = a_0 < a_1 < \dots < a_N = K$ with $a_{i+1} - a_i \leq \rho$ for $i = 0, \dots, N-1$, this enables us to find ε_0 such that for all $\varepsilon < \varepsilon_0$, every $i = 0, \dots, N$ and all $\lambda \in [K^{-2}, 1]$,

$$|u_\varepsilon^\beta(\lambda a_i, \lambda K) - u^\beta(\lambda a_i, \lambda K)| \leq \frac{\delta}{2}. \quad (39)$$

Now, let $(a, b) \in [K^{-1}, K]$ be arbitrary, but to fix the ideas, assume without loss of generality that $a \leq b$. Since $u_\varepsilon^\beta(a, b)$ is increasing in a on $a \leq b$,

$$u_\varepsilon^\beta(a, b) \in [u_\varepsilon^\beta(a_i \frac{b}{K}, b), u_\varepsilon^\beta(a_{i+1} \frac{b}{K}, b)],$$

where i is such that $a_i \leq a \frac{K}{b} \leq a_{i+1}$, and by the property (39), also

$$u_\varepsilon^\beta(a, b) \in [u^\beta(a_i \frac{b}{K}, b) - \frac{\delta}{2}, u^\beta(a_{i+1} \frac{b}{K}, b) + \frac{\delta}{2}].$$

We finally use the uniform continuity of u^β to conclude that $u_\varepsilon^\beta(a, b) \in [u^\beta(a, b) - \delta, u^\beta(a, b) + \delta]$. \square

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